

BCS-BEC crossover in 3D Fermi gases with spherical spin-orbit coupling

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We present a systematic theoretical study of the BCS-BEC crossover problem in 3D atomic Fermi gases at zero temperature with a spherical spin-orbit coupling which can be realized by a synthetic non-Abelian gauge field coupled to fermions. Our investigations are based on the path integral formalism which is a powerful theoretical scheme for the study of the properties of bound state, the superfluid ground state, and the collective excitations in the BCS-BEC crossover. At large spin-orbit coupling, the system enters the BEC state of a novel type of bound state (referred to as rashbon) which possesses a non-trivial effective mass. Analytical results and interesting universal behaviors for various physical quantities at large spin-orbit coupling are obtained. Our theoretical predictions can be tested in future experiments of cold Fermi gases with 3D spherical spin-orbit coupling.

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I. INTRODUCTION

It has been widely accepted for a long time that, by tuning the attractive strength in a Fermi gas, one can realize a smooth crossover from the Bardeen–Cooper–Schrieffer (BCS) superfluidity at weak attraction to Bose–Einstein condensation (BEC) of difermion molecules at strong attraction [1–9]. For a dilute Fermi gas in three dimensions where the effective range r_0 of the short range interaction is much smaller than the inter-particle distance, the system can be characterized by a dimensionless parameter $1/(k_F a_s)$, where a_s is the s-wave scattering length of the short range interaction and k_F is the Fermi momentum in the absence of interaction. The BCS-BEC crossover occurs when the parameter $1/(k_F a_s)$ is tuned from negative to positive values, and the BCS and BEC limits correspond to the cases $1/(k_F a_s) \rightarrow -\infty$ and $1/(k_F a_s) \rightarrow +\infty$, respectively. The BCS-BEC crossover has also become an interesting issue for the studies of dense nuclear and quark matter which may exists in the core of compact stars [10–12].

This BCS-BEC crossover phenomenon has been successfully demonstrated in ultracold fermionic atoms, where the s-wave scattering length and hence the parameter $1/(k_F a_s)$ was tuned by means of the Feshbach resonance [13–15]. At the resonant point or the so-called unitary point where $a_s \rightarrow \infty$, the only length scale of the system is the inter-particle distance ($\sim k_F^{-1}$). Therefore, the properties of the system at the unitary point $1/(k_F a_s) = 0$ become universal, i.e., independent of the details of the interactions. All physical quantities, scaled by their counterparts for the non-interacting Fermi gases, become universal constants. Determining these universal constants has been one of the most intriguing topics in the research of the cold Fermi gases [16, 17].

While the BCS-BEC crossover triggered by tuning the attraction strength between fermions from weak to strong ($1/(k_F a_s)$ from $-\infty$ to $+\infty$) has been comprehensively stud-

ied both theoretically and experimentally, it is always interesting to look for other mechanisms to realize the BCS-BEC crossover. Recent experimental breakthrough in generating synthetic non-Abelian gauge field has opened up the opportunity to study the spin-orbit coupling (SOC) effect in cold atomic gases [18]. For fermionic atoms [19], it may provide an alternative way to realize the BCS-BEC crossover [20].

Such SOC for fermions can be realized by generating a synthetic SU(2) gauge field A^μ of the type $A_i^\mu = \lambda_i \delta_i^\mu$ [21]. From the minimum coupling scheme, the resulting Hamiltonian for fermions reads $\mathcal{H} = \mathbf{p}^2/(2m) - \boldsymbol{\tau} \cdot \boldsymbol{\xi}_\lambda$ where $\boldsymbol{\xi}_\lambda = (\lambda_x p_x, \lambda_y p_y, \lambda_z p_z)$. The gauge field strengths λ_i ($i = x, y, z$) characterize the spin-orbital coupling constants. The problem of the difermion bound state in 3D case in the presence of SOC has been studied in [21]. Three special cases were considered: (1) $\lambda_x = \lambda_y = 0$ and $\lambda_z = \lambda$ (called extreme prolate (EP)); (2) $\lambda_x = \lambda_y = \lambda$ and $\lambda_z = 0$ (called extreme oblate (EO)); (3) $\lambda_x = \lambda_y = \lambda_z = \lambda$ (called spherical (S)). The EO-type SOC is physically equivalent to the Rashba SOC which is interesting for condensed matter physics. For EO- and S-type SOC, it was shown that the difermion bound state exists even for $a_s < 0$ where the bound state does not exist in the absence of SOC. With increased SOC, the bounding energy is generally enhanced [21]. The bound state also possesses a non-trivial effective mass which is generally larger than twice of the fermion mass m [22–24]. Such a novel bound state caused by the SOC is now referred to as rashbons in the literatures [22]. For the 2D case, the bound state exists for arbitrarily small attraction. It was shown in [25] that the EO-type SOC or the Rashba SOC enhances the binding energy and the bound state also has a non-trivial effective mass. This is analogous to the catalysis of the dynamical mass generation by an external non-Abelian gauge field in quantum field theory [26].

Because of the presence of novel bound state with SOC, it has been proposed that a dilute Fermi gas with EO- and S-type SOC can undergo a smooth crossover from the BCS superfluid state to the Bose-Einstein condensation of rashbons (RBEC) even for negative values of $1/(k_F a_s)$ if the SOC constant λ is tuned from small to large values [20]. Due to the presence of SOC constant λ , the 3D BCS-BEC

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crossover problem depends on two dimensionless parameters: $1/(k_F a_s)$ and λ/k_F (we set $m = 1$ in this paper). The BCS-BEC crossover problem and anisotropic superfluidity in 3D Fermi gases with EO-type SOC has been extensively studied [23, 27]. It was shown that the system enters the RBEC regime at $\lambda/k_F \sim 1$ for EO-type SOC for negative values of $1/(k_F a_s)$. The BCS-BEC crossover in 2D Fermi gases with EO-type SOC was also studied [25, 28]. Similar conclusions were found for the 2D case.

In this paper, we present a systematic theoretical study of the BCS-BEC crossover in 3D Fermi gases at zero temperature with S-type SOC. Especially, we will study the properties of the collective modes along the BCS-BEC crossover and the effective interaction among the rashbons in the RBEC regime. As far as we know, in the presence of SOC, these two interesting issues has not yet been studied (See the *note added*). For S-type SOC, the superfluid ground state is isotropic, which brings much convenience to the computations, and enables us to obtain various analytical results and universal behaviors at large SOC.

This paper is organized as follows. In Section II, we set up the functional path integral formalism for the BCS-BEC crossover problem with a spherical SOC. Then we first determine the binding energy and the effective mass of the rashbon at vanishing density and temperature (the vacuum in the presence of SOC) in Section III. The ground state properties, such as the solution of the gap and number equations, fermion momentum distribution, the condensate fraction, and the superfluid density are discussed in Section IV. We derive the Gross-Pitaevskii free energy for the weakly interacting rashbon condensate at large SOC and determine the rashbon-rashbon scattering length in Section V. The properties of the collective excitations, such as the gapless Goldstone mode and the massive Anderson-Higgs mode, are investigated in Section VI. We summarize in Section VII.

II. MODEL AND EFFECTIVE POTENTIAL

The Hamiltonian describing spin-1/2 fermions moving in three spatial dimensions in a SU(2) non-Abelian gauge field \mathbf{A}^μ is given by

$$H_{\text{GF}} = \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) \frac{(\hat{\mathbf{p}} - \mathbf{A}^\mu \tau_\mu)^2}{2m} \psi(\mathbf{r}) \quad (1)$$

where $\psi(\mathbf{r}) = [\psi_\uparrow(\mathbf{r}), \psi_\downarrow(\mathbf{r})]^T$ represents the two-component fermion fields and $\hat{\mathbf{p}} = -i\hbar\nabla$ is the momentum operator. Here τ_μ are Pauli spin operators ($\mu = x, y, z$). In the following we use the natural units $\hbar = k_B = m = 1$. Considering a spherical uniform gauge field, $A_i^\mu = -\lambda \delta_i^\mu$ ($i = x, y, z$), the Hamiltonian can be reduced to

$$H_{\text{GF}} = \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) \left(\frac{\hat{\mathbf{p}}^2}{2} + \lambda \boldsymbol{\tau} \cdot \hat{\mathbf{p}} \right) \psi(\mathbf{r}), \quad (2)$$

which can be regarded as a generalized Rashba Hamiltonian with an isotropic SOC. Here the sign of the gauge field

strength λ is not important, since the physical quantities depend only on the parameter λ^2 as we will show in this paper. In this paper, we set $\lambda > 0$ without loss of generality.

We now turn on a short-range attractive interaction in the spin-singlet channel. For homogeneous Fermi gases, we define the Fermi momentum k_F through the fermion density $n = N/V = k_F^3/(3\pi^2)$, and the Fermi energy is $\epsilon_F = k_F^2/2$. In the dilute limit $k_F r_0 \ll 1$ (r_0 — effective range of the interaction), the interaction Hamiltonian can be modeled by a contact interaction. The total Hamiltonian of the system can be written as

$$H = \int d^3\mathbf{r} \psi^\dagger(\mathbf{r}) (\mathcal{H}_0 + \mathcal{H}_{\text{so}}) \psi(\mathbf{r}) - U \int d^3\mathbf{r} \psi_\uparrow^\dagger(\mathbf{r}) \psi_\downarrow^\dagger(\mathbf{r}) \psi_\downarrow(\mathbf{r}) \psi_\uparrow(\mathbf{r}), \quad (3)$$

where $\mathcal{H}_0 = \hat{\mathbf{p}}^2/2 - \mu$ is the free single-particle Hamiltonian with μ being the chemical potential, $\mathcal{H}_{\text{so}} = \lambda \boldsymbol{\tau} \cdot \hat{\mathbf{p}}$ is the SOC term, and $U > 0$ denotes the attractive s-wave interaction between unlike spins. For the validity of such a contact interaction, another dilute condition $\lambda r_0 \ll 1$ should be satisfied [29]. We also note that the Galilean invariance of the Hamiltonian in the absence of SOC ($\lambda = 0$) is broken by the SOC term. However, as it will be shown, the Galilean invariance can be recovered at the boson (rashbon) level for large λ .

In the functional path integral formalism, the partition function of the system is

$$\mathcal{Z} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \exp \{-S[\psi, \bar{\psi}]\}, \quad (4)$$

where

$$S[\psi, \bar{\psi}] = \int_0^\beta d\tau \int d^3\mathbf{r} \bar{\psi} \partial_\tau \psi + \int_0^\beta d\tau H(\psi, \bar{\psi}). \quad (5)$$

Here $\beta = 1/T$ and $H(\psi, \bar{\psi})$ is obtained by replacing the field operators ψ^\dagger and ψ with the Grassmann variables $\bar{\psi}$ and ψ , respectively. To decouple the interaction term we introduce the auxiliary complex pairing field $\Phi(x) = -U\psi_\downarrow(x)\psi_\uparrow(x)$ [$x = (\tau, \mathbf{r})$] and apply the Hubbard-Stratonovich transformation. Using the 4-component Nambu-Gor'kov spinor $\Psi(x) = [\psi_\uparrow, \psi_\downarrow, \bar{\psi}_\uparrow, \bar{\psi}_\downarrow]^T$, we express the partition function as

$$\mathcal{Z} = \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \mathcal{D}\Phi \mathcal{D}\Phi^* \exp \left\{ -\frac{1}{U} \int dx |\Phi(x)|^2 + \frac{1}{2} \int dx \int dx' \bar{\Psi}(x) \mathbf{G}^{-1}(x, x') \Psi(x') \right\}, \quad (6)$$

where the inverse single-particle Green function $\mathbf{G}^{-1}(x, x')$ is given by

$$\mathbf{G}^{-1} = \begin{pmatrix} -\partial_\tau - \mathcal{H}_0 - \mathcal{H}_{\text{so}} & i\tau_y \Phi(x) \\ -i\tau_y \Phi^*(x) & -\partial_\tau + \mathcal{H}_0 - \mathcal{H}_{\text{so}}^* \end{pmatrix} \delta(x - x'). \quad (7)$$

Integrating out the fermion fields, we obtain $\mathcal{Z} = \int \mathcal{D}\Phi \mathcal{D}\Phi^* \exp \{-S_{\text{eff}}[\Phi, \Phi^*]\}$, where the effective action reads

$$S_{\text{eff}}[\Phi, \Phi^*] = \frac{1}{U} \int dx |\Phi(x)|^2 - \frac{1}{2} \text{Tr} \ln [\mathbf{G}^{-1}(x, x')]. \quad (8)$$

III. TWO-BODY PROBLEM

In this section, we study the two-body problem at vanishing density. We will determine the binding energy and effective mass of difermion bound state formed in the non-Abelian gauge field. The systematic way to study the two-body problem in presence of a nonzero spin-orbit coupling λ is to consider the Green function $\Gamma(Q)$ of the fermion pairs, where $Q = (i\nu_n, \mathbf{q})$ with $\nu_n = 2n\pi T$ (n integer) being the bosonic Matsubara frequency. For zero density, we need to consider the case $\Phi = 0$. In the functional path integral formalism, $\Gamma^{-1}(Q)$ can be obtained from its coordinate representation defined as

$$\Gamma^{-1}(x, x') = \frac{1}{\beta V} \frac{\delta^2 \mathcal{S}_{\text{eff}}[\Phi, \Phi^*]}{\delta \Phi^*(x) \delta \Phi(x')} \Big|_{\Phi=0}. \quad (9)$$

For $\Phi = 0$, the single-particle Green function $\mathbf{G}(K)$ reduces to its non-interacting form

$$\mathcal{G}_0(K) = \begin{pmatrix} g_+(K) & 0 \\ 0 & g_-(K) \end{pmatrix}, \quad (10)$$

where $K = (i\omega_n, \mathbf{k})$ with $\omega_n = (2n+1)\pi T$ being the fermionic Matsubara frequency. Here the elements $g_{\pm}(K)$ read

$$g_+(K) = \frac{1}{i\omega_n - \xi_{\mathbf{k}} - \xi_{\text{so}}}, \quad (11)$$

$$g_-(K) = \frac{1}{i\omega_n + \xi_{\mathbf{k}} - \xi_{\text{so}}^*}.$$

Here $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ with $\epsilon_{\mathbf{k}} = \mathbf{k}^2/2$, $\xi_{\text{so}} = \lambda(\tau_x k_x + \tau_y k_y + \tau_z k_z)$ and $\xi_{\text{so}}^* = \lambda(\tau_x k_x - \tau_y k_y + \tau_z k_z)$. The inverse in $g_{\pm}(K)$ can be worked out and we obtain

$$g_+(K) = \frac{i\omega_n - \xi_{\mathbf{k}} + \xi_{\text{so}}}{(i\omega_n - \xi_{\mathbf{k}})^2 - \lambda^2 \mathbf{k}^2}, \quad (12)$$

$$g_-(K) = \frac{i\omega_n + \xi_{\mathbf{k}} + \xi_{\text{so}}^*}{(i\omega_n + \xi_{\mathbf{k}})^2 - \lambda^2 \mathbf{k}^2}.$$

The single-particle excitation spectrum therefore has two branches, $\xi_{\mathbf{k}}^{\pm} = \xi_{\mathbf{k}} \pm \lambda|\mathbf{k}|$, due to the spin-orbit coupling.

Using the free fermion propagators $g_{\pm}(K)$, $\Gamma^{-1}(Q)$ can be expressed as

$$\Gamma^{-1}(Q) = \frac{1}{U} - \frac{1}{2} \sum_K \text{Tr} [g_+(K+Q) \tau_y g_-(K) \tau_y]. \quad (13)$$

Completing the Matsubara frequency sum and making the analytical continuation $i\nu_n \rightarrow \omega + i0^+$, the real part of $\Gamma^{-1}(\omega + i0^+, \mathbf{q})$ takes the form

$$\begin{aligned} \Gamma_{\text{R}}^{-1}(\omega, \mathbf{q}) &\equiv \text{Re} \Gamma^{-1}(\omega + i0^+, \mathbf{q}) \\ &= \frac{1}{U} - \frac{1}{4} \sum_{\alpha, \gamma=\pm} \sum_{\mathbf{k}} \frac{1 - f(\xi_{\mathbf{k}+\mathbf{q}/2}^{\alpha}) - f(\xi_{\mathbf{k}-\mathbf{q}/2}^{\gamma})}{\xi_{\mathbf{k}+\mathbf{q}/2}^{\alpha} + \xi_{\mathbf{k}-\mathbf{q}/2}^{\gamma} - \omega} (1 + \alpha\gamma \mathcal{T}_{\mathbf{k}\mathbf{q}}), \end{aligned} \quad (14)$$

where $f(E) = 1/(e^{\beta E} + 1)$ is the Fermi-Dirac distribution function, and $\mathcal{T}_{\mathbf{k}\mathbf{q}}$ is defined as

$$\mathcal{T}_{\mathbf{k}\mathbf{q}} = \frac{(\mathbf{k} + \mathbf{q}/2) \cdot (\mathbf{k} - \mathbf{q}/2)}{|\mathbf{k} + \mathbf{q}/2| |\mathbf{k} - \mathbf{q}/2|}. \quad (15)$$

We use the notations $\sum_K = T \sum_n \sum_{\mathbf{k}}$ and $\sum_{\mathbf{k}} = \int d^3\mathbf{k}/(2\pi)^3$ throughout this paper. Note that $\Gamma^{-1}(Q)$ takes the form similar to that of the relativistic systems [11], due to the fact that \mathcal{H}_{so} behaves like a Dirac Hamiltonian.

The integral over the fermion momentum \mathbf{k} is divergent and the contact coupling U needs to be regularized. For a short range interaction potential with its s-wave scattering length a_s , it is natural to regularize U by means of the two-body problem in the absence of SOC. We have

$$\frac{1}{U} = -\frac{1}{4\pi a_s} + \sum_{\mathbf{k}} \frac{1}{2\epsilon_{\mathbf{k}}}. \quad (16)$$

For the pure two-body problem at vanishing density and temperature, we discard the Fermi-Dirac distribution function. The energy-momentum dispersion $\omega_{\mathbf{q}}$ of the pair excitation is defined as the solution $\omega + 2\mu = \omega_{\mathbf{q}}$ of the two-body equation $\Gamma_{\text{R}}^{-1}(\omega, \mathbf{q}) = 0$. After some manipulations, the two-body equation becomes

$$\sum_{\mathbf{k}} \left(\frac{1}{\mathbf{k}^2} - \frac{\mathcal{E}_{\mathbf{k}\mathbf{q}}}{\mathcal{E}_{\mathbf{k}\mathbf{q}}^2 - 4\lambda^2 \mathbf{k}^2 - \frac{4\lambda^4 \mathbf{k}^2 \mathbf{q}^2 \sin^2 \varphi}{\mathcal{E}_{\mathbf{k}\mathbf{q}}^2 - \lambda^2 \mathbf{q}^2}} \right) = \frac{1}{4\pi a_s}. \quad (17)$$

Here φ is the angle between \mathbf{k} and \mathbf{q} , and $\mathcal{E}_{\mathbf{k}\mathbf{q}} = \epsilon_{\mathbf{k}+\mathbf{q}/2} + \epsilon_{\mathbf{k}-\mathbf{q}/2} - \omega_{\mathbf{q}} = \mathbf{k}^2 + \mathbf{q}^2/4 - \omega_{\mathbf{q}}$.

A. Bound State and Binding Energy

We are interested in whether there exist difermion bound state in the presence of SOC. For this purpose, we first consider zero center-of-mass momentum \mathbf{q} and determine the energy regime where the imaginary part of $\Gamma^{-1}(\omega + i0^+, \mathbf{q} = 0)$ vanishes. We have

$$\begin{aligned} \text{Im} \Gamma^{-1}(\omega + i0^+, \mathbf{q} = 0) \\ = -\frac{1}{4\pi} \sum_{\alpha=\pm} \int_0^{\infty} k^2 dk \delta(k^2 + 2\alpha\lambda k - \omega - 2\mu). \end{aligned} \quad (18)$$

Therefore, a bound state exists if the equation $\Gamma_{\text{R}}^{-1}(\omega, \mathbf{q} = 0) = 0$ has a solution in the regime $-\infty < \omega + 2\mu < -\lambda^2$.

The binding energy E_{B} in the presence of nonzero SOC is determined by the solution of $\omega + 2\mu = -E_{\text{B}}$ for the equation $\Gamma_{\text{R}}^{-1}(\omega, \mathbf{q} = 0) = 0$. From the imaginary part of $\Gamma^{-1}(\omega + i0^+, \mathbf{q} = 0)$, the binding energy E_{B} must be larger than a threshold $E_{\text{th}} = \lambda^2$. The equation determining E_{B} reads

$$\int_0^{\infty} k^2 dk \left[\frac{1}{k^2} - \frac{k^2 + E_{\text{B}}}{(k^2 + E_{\text{B}})^2 - 4\lambda^2 k^2} \right] = \frac{\pi}{2a_s}. \quad (19)$$

Completing the integrals analytically, we obtain a simple algebraic equation for E_{B} ,

$$\frac{E_{\text{B}} - 2\lambda^2}{\sqrt{E_{\text{B}} - \lambda^2}} = \frac{1}{a_s}. \quad (20)$$

We find that, for arbitrary scattering length a_s , there always exists a solution $E_{\text{B}} > \lambda^2$. Therefore, the difermion bound

state can form in the presence of SOC even for $a_s < 0$ where no bound state exists in the absence of SOC.

The solution of Eq. (20) can be analytically expressed as

$$E_B = \lambda^2 + \frac{1}{4} \left(\frac{1}{a_s} + \sqrt{\frac{1}{a_s^2} + 4\lambda^2} \right)^2. \quad (21)$$

Therefore, the quantity E_B/λ^2 depends only on the dimensionless parameter $\kappa = 1/(\lambda a_s)$. We have

$$\frac{E_B}{\lambda^2} = \mathcal{J}(\kappa), \quad (22)$$

where the function $\mathcal{J}(\kappa)$ is defined as

$$\mathcal{J}(\kappa) = 1 + \frac{1}{4} \left(\kappa + \sqrt{\kappa^2 + 4} \right)^2. \quad (23)$$

We are interested in the case $\lambda a_s \rightarrow \infty$ or $\kappa = 0$. This happens when $a_s \rightarrow \infty$ (unitary point of the Feshbach resonance) for fixed λ or $\lambda \rightarrow \infty$ for fixed a_s . In this case, we have $\mathcal{J} = 2$ and a very simple result

$$E_B(\lambda a_s \rightarrow \infty) = 2\lambda^2. \quad (24)$$

In general, the numerical result for the quantity $E_B/\lambda^2 - 1 = \mathcal{J}(\kappa) - 1$ is shown in Fig. 1.

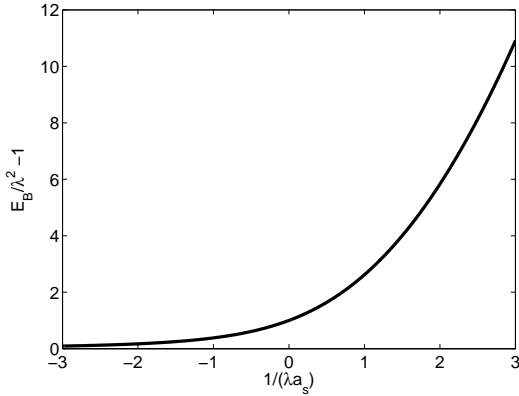


FIG. 1: The quantity $E_B/\lambda^2 - 1$ as a function of the dimensionless parameter $\kappa = 1/(\lambda a_s)$.

B. Molecule Effective Mass

For small nonzero center-of-mass momentum \mathbf{q} , the solution for $\omega_{\mathbf{q}}$ can be written as $\omega_{\mathbf{q}} = -E_B + \mathbf{q}^2/(2m_B)$, where m_B is referred to as the effective mass of the bound state. Substituting this dispersion into the equation $\Gamma_R^{-1}(\omega, \mathbf{q}) = 0$ and expanding the equation to the order $O(\mathbf{q}^2)$, we obtain

$$\begin{aligned} & \left(1 - \frac{2m}{m_B} \right) \int_0^\infty k^2 dk \frac{(k^2 + E_B)^2 + 4\lambda^2 k^2}{[(k^2 + E_B)^2 - 4\lambda^2 k^2]^2} \\ &= \frac{4}{3} \int_0^\infty k^2 dk \frac{8\lambda^4 k^2}{(k^2 + E_B)[(k^2 + E_B)^2 - 4\lambda^2 k^2]^2}. \end{aligned} \quad (25)$$

Defining a new variable $x = k/\lambda$, this equation becomes

$$\begin{aligned} & \left(1 - \frac{2m}{m_B} \right) \int_0^\infty dx x^2 \frac{(x^2 + \mathcal{J})^2 + 4x^2}{[(x^2 + \mathcal{J})^2 - 4x^2]^2} \\ &= \frac{4}{3} \int_0^\infty dx x^2 \frac{8x^2}{(x^2 + \mathcal{J})[(x^2 + \mathcal{J})^2 - 4x^2]^2}. \end{aligned} \quad (26)$$

Completing the integrals analytically, we obtain

$$\frac{2m}{m_B} = \frac{7}{3} - \frac{4}{3} \left(\frac{\mathcal{J} - 1}{\mathcal{J}} \right)^{3/2} - \frac{2}{\mathcal{J}}. \quad (27)$$

The effective mass therefore depends only on the combined parameter $\kappa = 1/(\lambda a_s)$. We have

$$\begin{aligned} \frac{2m}{m_B} &= \frac{7}{3} - \frac{4}{\kappa^2 + 4 + \kappa \sqrt{\kappa^2 + 4}} \\ &\quad - \frac{4}{3} \left(1 - \frac{2}{\kappa^2 + 4 + \kappa \sqrt{\kappa^2 + 4}} \right)^{3/2}. \end{aligned} \quad (28)$$

The numerical result for $m_B/2m$ is shown in Fig. 2. We find analytically that $m_B \rightarrow 2m$ in the limit $\kappa \rightarrow +\infty$ and $m_B \rightarrow 6m$ in the limit $\kappa \rightarrow -\infty$. For the case $\lambda a_s \rightarrow \infty$ or $\kappa = 0$, the effective mass reads

$$\frac{m_B(\lambda a_s \rightarrow \infty)}{2m} = \frac{3(4 + \sqrt{2})}{14} = 1.16. \quad (29)$$

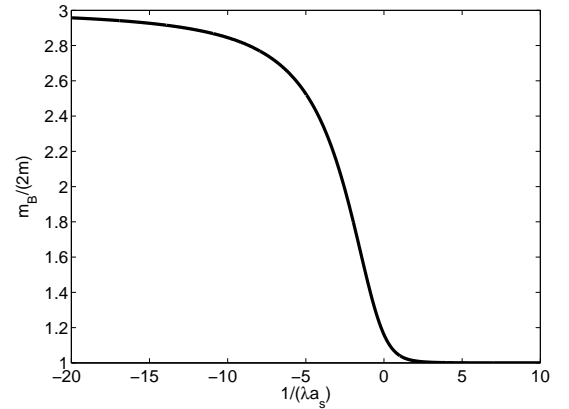


FIG. 2: The molecule effective mass m_B (divided by $2m$) as a function of the dimensionless parameter $\kappa = 1/(\lambda a_s)$.

In summary, the difermion bound state forms in the presence of SOC for arbitrary value of s-wave scattering length a_s . The bound state possesses a non-trivial binding energy E_B and a non-trivial effective mass $m_B > 2m$. Such type of bound state are referred to as rashbon in the previous literatures. Due to the formation of bound state at the BCS side of the resonance ($a_s < 0$) and the enhancement of binding energy in the presence of SOC, we expect that there will be a crossover from the BCS superfluid state to the Bose-Einstein condensation of rashbons if the spin-orbit coupling λ can be tuned from small to large values.

IV. SUPERFLUID GROUND STATE: MEAN FIELD THEORY

For the many-body problem, we first consider the properties of the superfluid ground state ($T = 0$) in the self-consistent mean-field theory. In the superfluid ground state, the pairing field $\Phi(x)$ acquires a nonzero expectation value $\langle \Phi(x) \rangle = \Delta$, which serves as the order parameter of the superfluidity. Without loss of generality, we set Δ to be real. Then we can express the pairing field as $\Phi(x) = \Delta + \phi(x)$, where $\phi(x)$ is the fluctuation around the mean field. The effective action $\mathcal{S}_{\text{eff}}[\Phi, \Phi^*]$ can be expanded in powers of the fluctuation,

$$\mathcal{S}_{\text{eff}}[\Phi, \Phi^*] = \mathcal{S}_{\text{eff}}^{(0)}(\Delta) + \mathcal{S}_{\text{eff}}^{(2)}[\phi, \phi^*] + \dots, \quad (30)$$

where $\mathcal{S}_{\text{eff}}^{(0)}(\Delta) \equiv \mathcal{S}_{\text{eff}}[\Delta, \Delta]$ is the saddle-point or mean-field effective action with the superfluid order parameter determined by the saddle point condition $\partial \mathcal{S}_{\text{eff}}^{(0)} / \partial \Delta = 0$.

In the mean-field approximation, the grand potential $\Omega = \mathcal{S}_{\text{eff}}[\Delta, \Delta] / (\beta V)$ can be expressed as

$$\Omega = \frac{\Delta^2}{U} - \frac{1}{2\beta} \sum_n \sum_{\mathbf{k}} \text{ln det} \mathcal{G}^{-1}(i\omega_n, \mathbf{k}), \quad (31)$$

where the inverse fermion Green function reads

$$\mathcal{G}^{-1}(i\omega_n, \mathbf{k}) = \begin{pmatrix} i\omega_n - \xi_{\mathbf{k}} - \xi_{\text{so}} & i\tau_y \Delta \\ -i\tau_y \Delta & i\omega_n + \xi_{\mathbf{k}} - \xi_{\text{so}}^* \end{pmatrix}. \quad (32)$$

Using the formula for block matrix, we first work out the determinate and obtain

$$\text{det} \mathcal{G}^{-1}(i\omega_n, \mathbf{k}) = [(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2][(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2], \quad (33)$$

where $E_{\mathbf{k}}^{\pm} = \sqrt{(\xi_{\mathbf{k}} \pm \lambda|\mathbf{k}|)^2 + \Delta^2}$ are quasiparticle excitation spectra. Then completing the Matsubara frequency sum we obtain

$$\Omega = \frac{\Delta^2}{U} + \sum_{\mathbf{k}} [\xi_{\mathbf{k}} - \mathcal{W}(E_{\mathbf{k}}^+) - \mathcal{W}(E_{\mathbf{k}}^-)], \quad (34)$$

where $\mathcal{W}(E) = E/2 + T \ln(1 + e^{-E/T})$. Note that the term $\sum_{\mathbf{k}} \xi_{\mathbf{k}} \equiv \frac{1}{2} \sum_{\mathbf{k}} (\xi_{\mathbf{k}}^+ + \xi_{\mathbf{k}}^-)$ is added to recover the correct ground state energy for the normal state ($\Delta = 0$).

A. Ground State Energy

At zero temperature, the ground state energy $E_G \equiv \Omega(T = 0)$ is $E_G = \Delta^2/U + (1/2) \sum_{\mathbf{k}} (2\xi_{\mathbf{k}} - E_{\mathbf{k}}^+ - E_{\mathbf{k}}^-)$. Using the fact that the binding energy E_B satisfies the equation

$$\frac{1}{U} = \frac{1}{2} \sum_{\alpha=\pm} \int_0^{\infty} \frac{k^2 dk}{2\pi^2} \frac{1}{k^2 + 2\alpha\lambda k + E_B}, \quad (35)$$

we can express the ground-state energy in terms of E_B as

$$E_G = \frac{1}{2} \sum_{\alpha=\pm} \int_0^{\infty} \frac{k^2 dk}{2\pi^2} \left(\frac{\Delta^2}{k^2 + 2\alpha\lambda k + E_B} - E_k^{\alpha} + \xi_k^{\alpha} \right). \quad (36)$$

Since the integral is convergent, we can use the trick $k^2 \pm 2\lambda k = (k \pm \lambda)^2 - \lambda^2$ and convert the integration variables to $k \pm \lambda$. Then we obtain

$$E_G = \int_0^{\infty} \frac{dk}{2\pi^2} (k^2 + \lambda^2) \left(\frac{\Delta^2}{k^2 + E_B - \lambda^2} - \tilde{E}_k + \tilde{\xi}_k \right), \quad (37)$$

where

$$\tilde{\xi}_k = \epsilon_k - \tilde{\mu}, \quad \tilde{E}_k = \sqrt{(\epsilon_k - \tilde{\mu})^2 + \Delta^2} \quad (38)$$

with $\tilde{\mu} = \mu + \lambda^2/2$.

Using the above expression for E_G , the gap Δ and the chemical potential μ can be determined by $\partial E_G / \partial \Delta = 0$ and $\partial E_G / \partial \mu = -n$, i.e.,

$$\begin{aligned} \int_0^{\infty} dk (k^2 + \lambda^2) \left[\frac{1}{k^2 + E_B - \lambda^2} - \frac{1}{2\sqrt{(\epsilon_k - \tilde{\mu})^2 + \Delta^2}} \right] &= 0, \\ \int_0^{\infty} dk (k^2 + \lambda^2) \left[1 - \frac{\epsilon_k - \tilde{\mu}}{\sqrt{(\epsilon_k - \tilde{\mu})^2 + \Delta^2}} \right] &= 2\pi^2 n \end{aligned} \quad (39)$$

We notice that the above expressions for the gap and number equations can be analytically evaluated using the elliptic functions, like the analytical treatment for the gap and number equations in the absence of SOC [30].

B. Fermion Green Function

The explicit form of the fermion Green function $\mathcal{G}(i\omega_n, \mathbf{k})$ can be evaluated using the formula for block matrix. In the Nambu-Gor'kov space, it takes the form

$$\mathcal{G}(i\omega_n, \mathbf{k}) = \begin{pmatrix} \mathcal{G}_{11}(i\omega_n, \mathbf{k}) & \mathcal{G}_{12}(i\omega_n, \mathbf{k}) \\ \mathcal{G}_{21}(i\omega_n, \mathbf{k}) & \mathcal{G}_{22}(i\omega_n, \mathbf{k}) \end{pmatrix}. \quad (40)$$

The matrix elements can be expressed as

$$\begin{aligned} \mathcal{G}_{11}(i\omega_n, \mathbf{k}) &= \mathcal{A}_{11}(i\omega_n, \mathbf{k}) \hat{I} + \mathcal{B}_{11}(i\omega_n, \mathbf{k}) \hat{M}, \\ \mathcal{G}_{22}(i\omega_n, \mathbf{k}) &= \mathcal{A}_{22}(i\omega_n, \mathbf{k}) \hat{I} + \mathcal{B}_{22}(i\omega_n, \mathbf{k}) \hat{M}^*, \\ \mathcal{G}_{12}(i\omega_n, \mathbf{k}) &= -i\tau_y [\mathcal{A}_{12}(i\omega_n, \mathbf{k}) \hat{I} + \mathcal{B}_{12}(i\omega_n, \mathbf{k}) \hat{M}^*], \\ \mathcal{G}_{21}(i\omega_n, \mathbf{k}) &= i\tau_y [\mathcal{A}_{21}(i\omega_n, \mathbf{k}) \hat{I} + \mathcal{B}_{21}(i\omega_n, \mathbf{k}) \hat{M}], \end{aligned} \quad (41)$$

where \hat{I} is the identity operator in the spin space and the operators \hat{M} and \hat{M}^* are defined as

$$\begin{aligned} \hat{M} &= \frac{\tau_x k_x + \tau_y k_y + \tau_z k_z}{|\mathbf{k}|}, \\ \hat{M}^* &= \frac{\tau_x k_x - \tau_y k_y + \tau_z k_z}{|\mathbf{k}|}. \end{aligned} \quad (42)$$

The explicit form of the quantities \mathcal{A}_{ij} and \mathcal{B}_{ij} are given by

$$\begin{aligned}\mathcal{A}_{11}(i\omega_n, \mathbf{k}) &= \frac{1}{2} \left[\frac{i\omega_n + \xi_{\mathbf{k}}^+}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2} + \frac{i\omega_n + \xi_{\mathbf{k}}^-}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} \right], \\ \mathcal{A}_{22}(i\omega_n, \mathbf{k}) &= \frac{1}{2} \left[\frac{i\omega_n - \xi_{\mathbf{k}}^+}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2} + \frac{i\omega_n - \xi_{\mathbf{k}}^-}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} \right], \\ \mathcal{A}_{12}(i\omega_n, \mathbf{k}) &= \frac{1}{2} \left[\frac{\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2} + \frac{\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} \right], \\ \mathcal{A}_{21}(i\omega_n, \mathbf{k}) &= \mathcal{A}_{12}(i\omega_n, \mathbf{k})\end{aligned}\quad (43)$$

and

$$\begin{aligned}\mathcal{B}_{11}(i\omega_n, \mathbf{k}) &= \frac{1}{2} \left[\frac{i\omega_n + \xi_{\mathbf{k}}^+}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2} - \frac{i\omega_n + \xi_{\mathbf{k}}^-}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} \right], \\ \mathcal{B}_{22}(i\omega_n, \mathbf{k}) &= -\frac{1}{2} \left[\frac{i\omega_n - \xi_{\mathbf{k}}^+}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2} - \frac{i\omega_n - \xi_{\mathbf{k}}^-}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} \right], \\ \mathcal{B}_{12}(i\omega_n, \mathbf{k}) &= -\frac{1}{2} \left[\frac{\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^+)^2} - \frac{\Delta}{(i\omega_n)^2 - (E_{\mathbf{k}}^-)^2} \right], \\ \mathcal{B}_{21}(i\omega_n, \mathbf{k}) &= -\mathcal{B}_{12}(i\omega_n, \mathbf{k}).\end{aligned}\quad (44)$$

Using the matrix elements of the Green function, we can calculate various quantities. First, the momentum distributions $n_{\uparrow}(\mathbf{k})$ and $n_{\downarrow}(\mathbf{k})$ for the two spin components can be evaluated as

$$\begin{aligned}n_{\uparrow}(\mathbf{k}) &\equiv \langle \bar{\psi}_{\mathbf{k}\uparrow} \psi_{\mathbf{k}\uparrow} \rangle \\ &= \frac{1}{\beta} \sum_n \left[\mathcal{A}_{11}(i\omega_n, \mathbf{k}) + \frac{k_z}{|\mathbf{k}|} \mathcal{B}_{11}(i\omega_n, \mathbf{k}) \right] e^{i\omega_n 0^+}, \\ n_{\downarrow}(\mathbf{k}) &\equiv \langle \bar{\psi}_{\mathbf{k}\downarrow} \psi_{\mathbf{k}\downarrow} \rangle \\ &= \frac{1}{\beta} \sum_n \left[\mathcal{A}_{11}(i\omega_n, \mathbf{k}) - \frac{k_z}{|\mathbf{k}|} \mathcal{B}_{11}(i\omega_n, \mathbf{k}) \right] e^{i\omega_n 0^+}.\end{aligned}\quad (45)$$

Second, the singlet and triplet pairing amplitudes can be expressed as

$$\begin{aligned}\phi_{\uparrow\downarrow}(\mathbf{k}) &\equiv \langle \psi_{\mathbf{k}\uparrow} \psi_{-\mathbf{k}\downarrow} \rangle \\ &= \frac{1}{\beta} \sum_n \left[-\mathcal{A}_{21}(i\omega_n, \mathbf{k}) + \frac{k_z}{k} \mathcal{B}_{21}(i\omega_n, \mathbf{k}) \right], \\ \phi_{\downarrow\uparrow}(\mathbf{k}) &\equiv \langle \psi_{\mathbf{k}\downarrow} \psi_{-\mathbf{k}\uparrow} \rangle \\ &= \frac{1}{\beta} \sum_n \left[\mathcal{A}_{21}(i\omega_n, \mathbf{k}) + \frac{k_z}{k} \mathcal{B}_{21}(i\omega_n, \mathbf{k}) \right], \\ \phi_{\uparrow\uparrow}(\mathbf{k}) &\equiv \langle \psi_{\mathbf{k}\uparrow} \psi_{-\mathbf{k}\uparrow} \rangle \\ &= -\frac{k_x - ik_y}{k} \frac{1}{\beta} \sum_n \mathcal{B}_{21}(i\omega_n, \mathbf{k}), \\ \phi_{\downarrow\downarrow}(\mathbf{k}) &\equiv \langle \psi_{\mathbf{k}\downarrow} \psi_{-\mathbf{k}\downarrow} \rangle \\ &= \frac{k_x + ik_y}{k} \frac{1}{\beta} \sum_n \mathcal{B}_{21}(i\omega_n, \mathbf{k}).\end{aligned}\quad (46)$$

Third, the gap equation for Δ can be expressed as

$$\Delta = -U \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \mathcal{A}_{12}(i\omega_n, \mathbf{k}).\quad (47)$$

C. Gap and Chemical Potential

Using the ground state energy E_G , the original forms of the gap and number equations at $T = 0$ are

$$\begin{aligned}\frac{1}{U} &= \frac{1}{2} \sum_{\mathbf{k}} \left(\frac{1}{2E_{\mathbf{k}}^+} + \frac{1}{2E_{\mathbf{k}}^-} \right), \\ n &= \sum_{\mathbf{k}} \left(1 - \frac{\xi_{\mathbf{k}}^+}{2E_{\mathbf{k}}^+} - \frac{\xi_{\mathbf{k}}^-}{2E_{\mathbf{k}}^-} \right).\end{aligned}\quad (48)$$

The pairing gap Δ and the chemical potential μ can be numerically solved for given values of $1/(k_F a_s)$ and λ/k_F . From now on, we denote the saddle point solution for the gap at zero temperature as Δ_0 . We also notice the relation

$$\frac{1}{\lambda a_s} = \frac{1}{k_F a_s} \left(\frac{\lambda}{k_F} \right)^{-1}.\quad (49)$$

(A) Analytical Results for Large SOC. We first obtain the analytical solution at large SOC, $\lambda/k_F \gg 1$. For large SOC, we expect $\mu < 0$ and $\Delta_0 \ll |\mu|$. Therefore, we can expand the equations in powers of $\Delta_0/|\mu|$ and keep only the leading order terms. The gap equation becomes

$$\frac{1}{U} = \frac{1}{2} \sum_{\alpha=\pm} \int_0^\infty \frac{k^2 dk}{2\pi^2} \frac{1}{k^2 + 2\alpha\lambda k - 2\mu} + O\left(\frac{\Delta_0^2}{|\mu|^2}\right),\quad (50)$$

Comparing with the two-body problem, we obtain

$$\mu \simeq -\frac{E_B}{2}.\quad (51)$$

Substituting this into the number equation, we obtain

$$\begin{aligned}n &= \frac{\Delta_0^2}{8\pi^2} \int_0^\infty k^2 dk \left[\frac{1}{(\xi_k^+)^2} + \frac{1}{(\xi_k^-)^2} \right] + O\left(\frac{\Delta_0^2}{|\mu|^2}\right) \\ &\simeq \frac{\Delta_0^2}{\pi^2} \int_0^\infty k^2 dk \frac{(k^2 + E_B)^2 + 4\lambda^2 k^2}{[(k^2 + E_B)^2 - 4\lambda^2 k^2]^2}.\end{aligned}\quad (52)$$

We notice that this integral also appears in Eq. (25). Completing the integral analytically, we obtain

$$\begin{aligned}\Delta_0^2 &\simeq 4\pi\lambda n \frac{(\mathcal{J} - 1)^{3/2}}{\mathcal{J}} \\ &= \frac{4\lambda [2\epsilon_F(\mathcal{J} - 1)]^{3/2}}{3\pi\mathcal{J}}.\end{aligned}\quad (53)$$

Therefore we have

$$\frac{\Delta_0}{\epsilon_F} \simeq \sqrt{\frac{16(\mathcal{J} - 1)^{3/2}}{3\pi} \frac{\lambda}{\mathcal{J} k_F}}.\quad (54)$$

In the limit $\lambda a_s \rightarrow \infty$, we have $\mathcal{J} = 2$ and therefore

$$\Delta_0^2(\lambda a_s \rightarrow \infty) \simeq 2\pi\lambda n = \frac{2\lambda(2\epsilon_F)^{3/2}}{3\pi}.\quad (55)$$

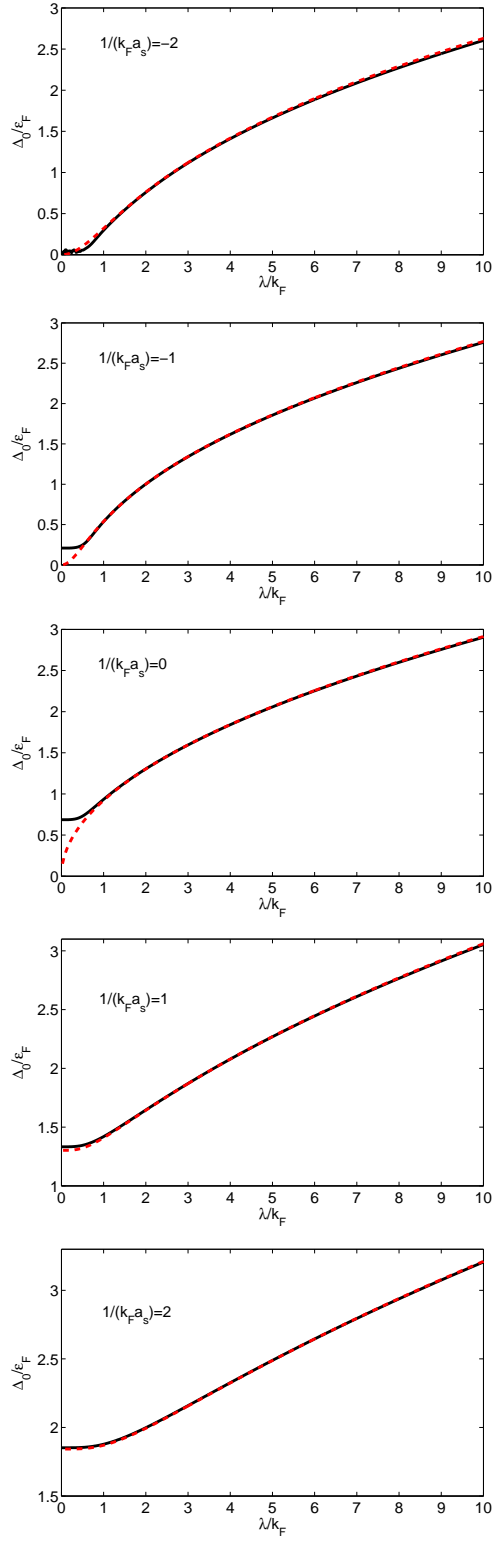


FIG. 3: The pairing gap Δ_0 (divided by ϵ_F) as a function of λ/k_F . The red dashed line shows the analytical result (54) or (60).

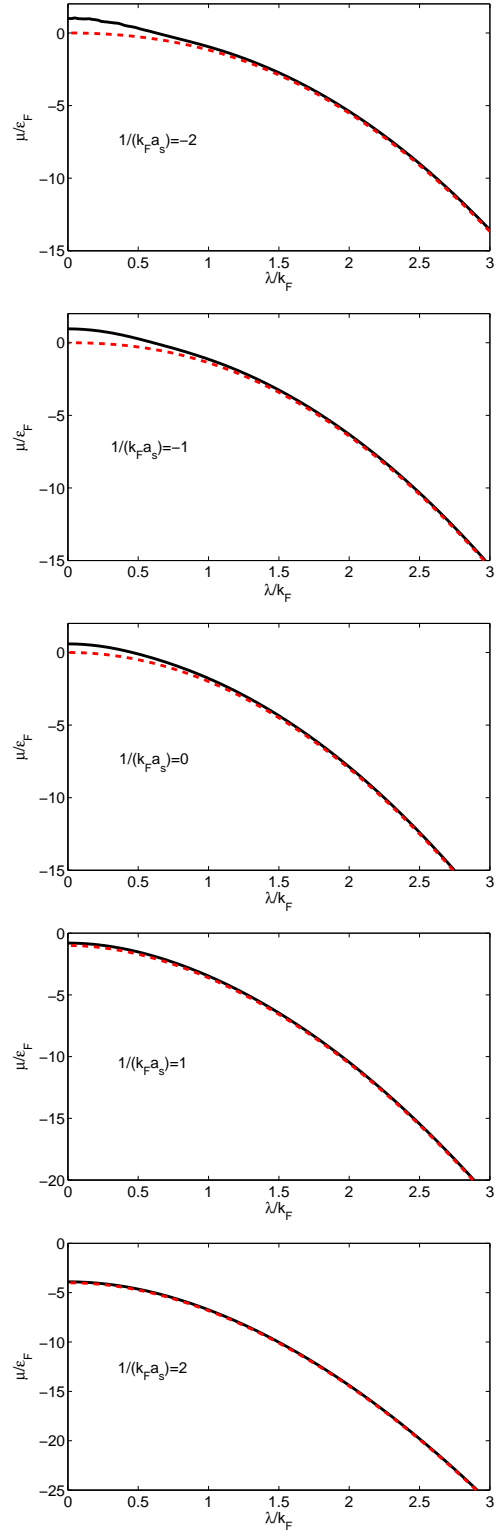


FIG. 4: The chemical potential μ (divided by ϵ_F) as a function of λ/k_F . The red dashed line shows the analytical result $\mu \simeq -\frac{E_B}{2}$ for large SOC.

It can be written as another interesting form

$$\frac{\Delta_0(\lambda a_s \rightarrow \infty)}{\epsilon_F} \simeq \sqrt{\frac{8}{3\pi}} \frac{\lambda}{k_F}. \quad (56)$$

Therefore, for very large SOC, the gap Δ_0 increases as $\Delta_0 \sim \sqrt{\lambda}$.

Beyond the leading order, we can write the chemical potential μ as

$$\mu = -\frac{E_B}{2} + \frac{\mu_B}{2}, \quad (57)$$

where $\mu_B = 2\mu + E_B \ll E_B$ is referred to as the effective chemical potential for bosons (rashbons). We will give an explicit expression for μ_B in Section V.

(B) Numerical Results. The gap and number equations (39) and (48) are equivalent. For numerical calculations, it is convenient to employ Eq. (39). If we define the following dimensionless quantities

$$g_1 = \frac{\lambda}{k_F}, \quad g_2 = \frac{1}{k_F a_s}, \quad x_1 = \frac{\mu}{\epsilon_F}, \quad x_2 = \frac{\Delta_0}{\epsilon_F}, \quad (58)$$

the gap and number equations can be written as the following dimensionless form

$$\begin{aligned} \int_0^\infty dz(z^2 + g_1^2) \left[\frac{1}{z^2 + g_1^2 \mathcal{J}(g_2/g_1) - g_1^2} - \frac{1}{\sqrt{(z^2 - g_1^2 - x_1)^2 + x_2^2}} \right] &= 0, \\ \int_0^\infty dz(z^2 + g_1^2) \left[1 - \frac{z^2 - g_1^2 - x_1}{\sqrt{(z^2 - g_1^2 - x_1)^2 + x_2^2}} \right] &= \frac{2}{3}. \end{aligned} \quad (59)$$

The integrals in the above equations can be analytically evaluated using elliptic functions [30]. For given values of g_1 and g_2 , these two equations determine x_1 and x_2 .

The numerical results are shown in Fig. 3 and Fig. 4. The red dashed lines correspond to the analytical results for large SOC,

$$\begin{aligned} x_1 &= -g_1^2 \mathcal{J}(g_2/g_1), \\ x_2 &= \sqrt{\frac{16g_1}{3\pi} \frac{[\mathcal{J}(g_2/g_1) - 1]^{3/2}}{\mathcal{J}(g_2/g_1)}}. \end{aligned} \quad (60)$$

We find that the pairing gap generally increases with increased λ/k_F . The numerical results become in good agreement with the analytical results when $\lambda/k_F \gtrsim 1$. Therefore, the system enters the rashbon BEC regime at $\lambda/k_F \sim 1$. For large positive value of $1/(k_F a_s)$, the analytical results are in good agreement with the numerical results even for small values of λ/k_F . For very large λ , we find the numerical results fit very well with the following scaling behavior

$$\frac{\Delta_0}{\epsilon_F} \simeq \sqrt{\frac{8}{3\pi}} \sqrt{\frac{\lambda}{k_F}}, \quad \frac{\mu}{\epsilon_F} \simeq -2 \left(\frac{\lambda}{k_F} \right)^2, \quad (61)$$

for both negative and positive values of $1/(k_F a_s)$.

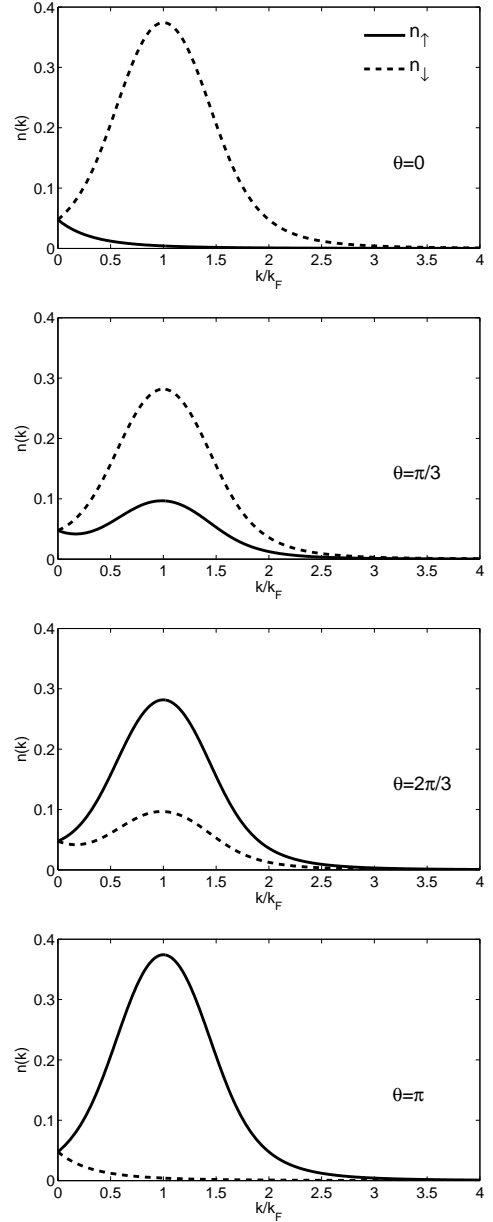


FIG. 5: The fermion momentum distributions $n_\uparrow(k)$ and $n_\downarrow(k)$ for various values of the polar angle θ . We set $1/(k_F a_s) = -1$ and $\lambda/k_F = 1$ in this calculation.

D. Fermion Momentum Distribution

From the matrix elements of the fermion Green function $\mathcal{G}(i\omega_n, \mathbf{k})$, we can obtain the momentum distributions $n_\uparrow(\mathbf{k})$ and $n_\downarrow(\mathbf{k})$ for the two spin components. The density of each component reads $n_\sigma = \sum_{\mathbf{k}} n_\sigma(\mathbf{k})$. We find that even though the density of the two components are the same, $n_\uparrow = n_\downarrow$, their distributions in the momentum space are different. At zero

temperature, their explicit expressions are given by

$$\begin{aligned} n_{\uparrow}(k, \theta) &= \frac{1}{4} \sum_{\alpha} \left(1 - \frac{\xi_k^{\alpha}}{E_k^{\alpha}} \right) + \frac{\cos \theta}{4} \sum_{\alpha} \alpha \left(1 - \frac{\xi_k^{\alpha}}{E_k^{\alpha}} \right), \\ n_{\downarrow}(k, \theta) &= \frac{1}{4} \sum_{\alpha} \left(1 - \frac{\xi_k^{\alpha}}{E_k^{\alpha}} \right) - \frac{\cos \theta}{4} \sum_{\alpha} \alpha \left(1 - \frac{\xi_k^{\alpha}}{E_k^{\alpha}} \right), \end{aligned} \quad (62)$$

where θ is the polar angle in the momentum space. We find that $n_{\uparrow}(\mathbf{k}) = n_{\downarrow}(\mathbf{k})$ only for $\theta = \pi/2$. We have $n_{\uparrow}(\mathbf{k}) < n_{\downarrow}(\mathbf{k})$ for $0 < \theta < \pi/2$ and $n_{\uparrow}(\mathbf{k}) > n_{\downarrow}(\mathbf{k})$ for $\pi/2 < \theta < \pi$.

In general, with increased SOC, the distribution broadens, which indicates a BCS-BEC crossover. A numerical example for $1/(k_F a_s) = -1$ and $\lambda/k_F = 1$ is shown in Fig. 5. The new feature here is that the distributions generally display non-monotonous behavior due to the SOC effect. We note that the peaks in the distributions are just located at $k = \lambda$.

E. Condensate Density

According to Leggett's definition [31], the condensate number of fermion pairs is given by

$$N_0 = \frac{1}{2} \sum_{\sigma_1, \sigma_2 = \uparrow, \downarrow} \int \int d^3 \mathbf{r}_1 d^3 \mathbf{r}_2 |\langle \psi_{\sigma_1}(\mathbf{r}_1) \psi_{\sigma_2}(\mathbf{r}_2) \rangle|^2. \quad (63)$$

For systems with only singlet pairing, this recovers the usual result $N_0 = \int \int d^3 \mathbf{r}_1 d^3 \mathbf{r}_2 |\langle \psi_{\uparrow}(\mathbf{r}_1) \psi_{\downarrow}(\mathbf{r}_2) \rangle|^2$ [32]. Converting this to the momentum space, we find that the condensate density $n_0 = N_0/V$ is a sum of all absolute squares of the pairing amplitudes,

$$\begin{aligned} n_0 &= \frac{1}{2} \sum_{\mathbf{k}} [|\phi_{\uparrow\downarrow}(\mathbf{k})|^2 + |\phi_{\downarrow\uparrow}(\mathbf{k})|^2 + |\phi_{\uparrow\uparrow}(\mathbf{k})|^2 + |\phi_{\downarrow\downarrow}(\mathbf{k})|^2] \\ &= \sum_{\mathbf{k}} \left\{ \left[\frac{1}{\beta} \sum_n \mathcal{A}_{21}(i\omega_n, \mathbf{k}) \right]^2 + \left[\frac{1}{\beta} \sum_n \mathcal{B}_{21}(i\omega_n, \mathbf{k}) \right]^2 \right\}. \end{aligned} \quad (64)$$

Completing the Matsubara frequency summation and taking the zero temperature limit, we obtain the explicit expression for $T = 0$,

$$\begin{aligned} n_0 &= \frac{\Delta_0^2}{16\pi^2} \int_0^{\infty} k^2 dk \left[\frac{1}{(E_k^+)^2} + \frac{1}{(E_k^-)^2} \right] \\ &= \frac{\Delta_0^2}{8\pi^2} \int_0^{\infty} dk \frac{k^2 + \lambda^2}{(\epsilon_k - \tilde{\mu})^2 + \Delta_0^2}. \end{aligned} \quad (65)$$

Generally, we can show that $n_0 < n/2$. For large SOC and/or attraction, we have $\Delta_0 \ll |\mu|$. Using the number equation (39) or (48) and expanding all terms in powers of $\Delta_0/|\mu|$, we find that

$$n_0 = \frac{n}{2} - O\left(\frac{\Delta_0^4}{|\mu|^4}\right). \quad (66)$$

Therefore, the condensate fraction $2N_0/N$ approaches unity at large SOC and/or attraction, as we expected.

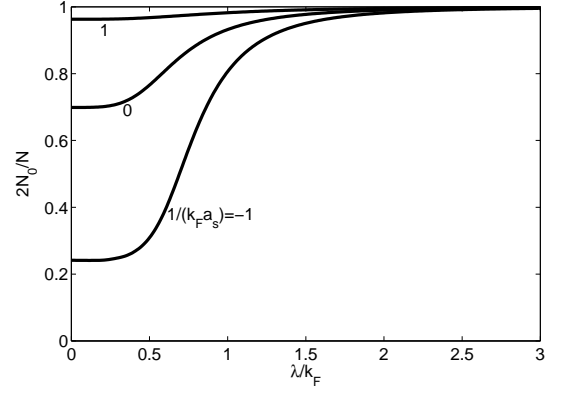


FIG. 6: The condensate fraction $2N_0/N$ as a function of λ/k_F for various values of $1/(k_F a_s)$.

In general, the condensate fraction $2N_0/N$ can be expressed as

$$\frac{2N_0}{N} = \frac{3x_2^2}{4} \int_0^{\infty} dz \frac{z^2 + g_1^2}{(z^2 - g_1^2 - x_1^2)^2 + x_2^2}. \quad (67)$$

It can be numerically obtained using the solutions of x_1 and x_2 from the gap and number equations. The numerical results are shown in Fig. 6. We find that, even for negative values of $1/(k_F a_s)$, the condensate fraction approaches unity around $\lambda/k_F \sim 2$. This is consistent with the observation from the solutions of the gap and number equations that the system enters the rashbon BEC regime at $\lambda/k_F \simeq 1$ for negative and small positive values of $1/(k_F a_s)$.

F. Superfluid Density

To evaluate the superfluid density n_s , we can employ the standard definition [33, 34]. When the superfluid moves with a uniform velocity $\mathbf{v}_s = (v_x, v_y, v_z)$, the superfluid order parameter transforms like $\Phi \rightarrow \Phi e^{2i\mathbf{q}_s \cdot \mathbf{r}}$ and $\Phi^* \rightarrow \Phi^* e^{-2i\mathbf{q}_s \cdot \mathbf{r}}$, where $\mathbf{q}_s = m\mathbf{v}_s$ ($m = 1$ in our units). The superfluid density n_s is defined as the response of the thermodynamic potential Ω to an infinitesimal velocity \mathbf{v}_s , i.e.,

$$\Omega(\mathbf{q}_s) = \Omega(\mathbf{0}) + \frac{1}{2} n_s \mathbf{q}_s^2 + O(\mathbf{q}_s^4). \quad (68)$$

The thermodynamic potential in the presence of a velocity \mathbf{v}_s can be evaluated by a gauge transformation for the fermion field $\psi \rightarrow \psi e^{-i\mathbf{q}_s \cdot \mathbf{r}}$. We have

$$\Omega(\mathbf{q}_s) = \frac{\Delta^2}{U} - \frac{1}{2\beta} \sum_n \sum_{\mathbf{k}} \text{ln det } \mathcal{G}_s^{-1}(i\omega_n, \mathbf{k}), \quad (69)$$

where the inverse fermion Green function in the presence of \mathbf{v}_s reads

$$\mathcal{G}_s^{-1}(i\omega_n, \mathbf{k}) = \mathcal{G}^{-1}(i\omega_n, \mathbf{k}) - \Sigma(\mathbf{q}_s). \quad (70)$$

Here the velocity-dependent part $\Sigma(\mathbf{q}_s)$ includes three parts, $\Sigma(\mathbf{q}_s) = \Sigma_1(\mathbf{q}_s) + \Sigma_2(\mathbf{q}_s) + \Sigma_3(\mathbf{q}_s)$, where

$$\begin{aligned}\Sigma_1(\mathbf{q}_s) &= \frac{1}{2}\mathbf{q}_s^2\sigma_3, \\ \Sigma_2(\mathbf{q}_s) &= \mathbf{k} \cdot \mathbf{q}_s\mathbb{1}, \\ \Sigma_3(\mathbf{q}_s) &= \lambda(\tau_x q_x \sigma_3 + \tau_y q_y \mathbb{1} + \tau_z q_z \sigma_3).\end{aligned}\quad (71)$$

Here σ_i ($i = 1, 2, 3$) and $\mathbb{1}$ are the Pauli matrices and the identity matrix in the Nambu-Gor'kov space, respectively. We note that the term $\Sigma_3(\mathbf{q}_s)$ is purely due to the presence of SOC.

(A) Derivation of the Superfluid Density. The superfluid density n_s can be obtained by the method of derivative expansion for $\Omega(\mathbf{q}_s)$, i.e.,

$$\Omega(\mathbf{q}_s) = \Omega(\mathbf{0}) + \frac{1}{2} \sum_n \frac{1}{n} \text{Tr} [\mathcal{G}\Sigma(\mathbf{q}_s)]^n. \quad (72)$$

We find that there are four types of nonzero contributions at the order $O(\mathbf{q}_s^2)$:

$$\begin{aligned}\Omega_1 &\sim \text{Tr}(\mathcal{G}\Sigma_1), & \Omega_2 &\sim \text{Tr}(\mathcal{G}\Sigma_2\mathcal{G}\Sigma_2), \\ \Omega_3 &\sim \text{Tr}(\mathcal{G}\Sigma_3\mathcal{G}\Sigma_3), & \Omega_4 &\sim \text{Tr}(\mathcal{G}\Sigma_2\mathcal{G}\Sigma_3).\end{aligned}\quad (73)$$

Since the superfluid state is isotropic, the superfluid density tensor should also be isotropic. We have carefully checked that all anisotropic terms vanish exactly. Completing the trace in the Nambu-Gor'kov and spin spaces, we finally obtain the following expressions for the four types of contributions:

$$\begin{aligned}\Omega_1 &= \frac{\mathbf{q}_s^2}{2} \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \frac{1}{2} (\mathcal{A}_{11} e^{i\omega_n 0^+} - \mathcal{A}_{22} e^{-i\omega_n 0^+}) \\ \Omega_2 &= \frac{\mathbf{q}_s^2}{2} \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \frac{\mathbf{k}^2}{3} (\mathcal{A}_{11}^2 + \mathcal{B}_{11}^2 + \mathcal{A}_{22}^2 + \mathcal{B}_{22}^2 + 2\mathcal{A}_{21}^2 + 2\mathcal{B}_{21}^2), \\ \Omega_3 &= \frac{\mathbf{q}_s^2}{2} \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \lambda^2 \left[(\mathcal{A}_{11}^2 + \mathcal{A}_{22}^2 + 2\mathcal{A}_{21}^2) - \frac{1}{3} (\mathcal{B}_{11}^2 + \mathcal{B}_{22}^2 + 2\mathcal{B}_{21}^2) \right], \\ \Omega_4 &= \frac{\mathbf{q}_s^2}{2} \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \frac{4\lambda|\mathbf{k}|}{3} (\mathcal{A}_{11}\mathcal{B}_{11} - \mathcal{A}_{22}\mathcal{B}_{22} + 2\mathcal{A}_{21}\mathcal{B}_{21}).\end{aligned}\quad (74)$$

Note that the first contribution is just from the total particle density n , $\Omega_1 = \frac{1}{2}n\mathbf{q}_s^2$. Collecting all terms, the superfluid density n_s is given by

$$\begin{aligned}n_s &= n + \frac{1}{\beta} \sum_n \sum_{\mathbf{k}} \left[\frac{\mathbf{k}^2}{3} (\mathcal{A}_{11}^2 + \mathcal{B}_{11}^2 + \mathcal{A}_{22}^2 + \mathcal{B}_{22}^2 + 2\mathcal{A}_{21}^2 + 2\mathcal{B}_{21}^2) + \frac{4\lambda|\mathbf{k}|}{3} (\mathcal{A}_{11}\mathcal{B}_{11} - \mathcal{A}_{22}\mathcal{B}_{22} + 2\mathcal{A}_{21}\mathcal{B}_{21}) \right. \\ &\quad \left. + \lambda^2 (\mathcal{A}_{11}^2 + \mathcal{A}_{22}^2 + 2\mathcal{A}_{21}^2) - \frac{\lambda^2}{3} (\mathcal{B}_{11}^2 + \mathcal{B}_{22}^2 + 2\mathcal{B}_{21}^2) \right].\end{aligned}\quad (75)$$

Completing the Matsubara frequency sum, we obtain the finite-temperature expression

$$\begin{aligned}n_s &= n - \int_0^\infty \frac{k^2 dk}{2\pi^2} \left[\frac{(k+\lambda)^2}{6} \frac{1}{2T} \frac{1}{\cosh^2\left(\frac{E_k^+}{2T}\right)} + \frac{(k-\lambda)^2}{6} \frac{1}{2T} \frac{1}{\cosh^2\left(\frac{E_k^-}{2T}\right)} \right] \\ &\quad - \frac{\lambda}{3} \int_0^\infty \frac{k dk}{2\pi^2} \left[\left(\xi_k^+ + \frac{\Delta^2}{\xi_k} \right) \frac{1 - 2f(E_k^+)}{E_k^+} - \left(\xi_k^- + \frac{\Delta^2}{\xi_k} \right) \frac{1 - 2f(E_k^-)}{E_k^-} \right].\end{aligned}\quad (76)$$

We have checked that this expression is consistent with the result for ordinary fermionic superfluids in the absence of SOC [33, 34]. Also, setting $\Delta = 0$, we find that $n_s(\Delta = 0)$ vanishes exactly.

We are interested in the zero temperature case. At zero temperature, the superfluid density reduces to

$$n_s = n - n_\lambda, \quad (77)$$

where n_λ is given by

$$n_\lambda = \frac{\lambda}{6\pi^2} \int_0^\infty k dk \left[\left(\xi_k^+ + \frac{\Delta_0^2}{\xi_k} \right) \frac{1}{E_k^+} - \left(\xi_k^- + \frac{\Delta_0^2}{\xi_k} \right) \frac{1}{E_k^-} \right]. \quad (78)$$

We notice that n_λ vanishes in the absence of SOC and we recover the usual result $n_s = n$ at $T = 0$ for ordinary fermionic superfluids [33, 34]. However, for nonzero SOC, n_λ is always positive and we have $n_s(\lambda \neq 0) < n$. Therefore, the SOC leads to suppression of the superfluid density.

(B) Analytical Result for Large SOC. To understand this interesting phenomenon, we first take a look at the large SOC limit. In this case we have $\mu \simeq -E_B/2$ and $\Delta \ll |\mu|$. Therefore, we can expand the expression in powers of $\Delta/|\mu|$ and keep only the leading order terms. Doing so, we obtain

$$\begin{aligned} n &\simeq \frac{\Delta_0^2}{8\pi^2} \int_0^\infty k^2 dk \left[\frac{1}{(\xi_k^+)^2} + \frac{1}{(\xi_k^-)^2} \right] \\ &\simeq \frac{\Delta_0^2}{\pi^2} \int_0^\infty k^2 dk \frac{(k^2 + E_B)^2 + 4\lambda^2 k^2}{[(k^2 + E_B)^2 - 4\lambda^2 k^2]^2} \end{aligned} \quad (79)$$

and

$$\begin{aligned} n_\lambda &\simeq \frac{\lambda \Delta_0^2}{6\pi^2} \int_0^\infty k dk \left\{ \frac{1}{\xi_k} \left(\frac{1}{\xi_k^+} - \frac{1}{\xi_k^-} \right) - \frac{1}{2} \left[\frac{1}{(\xi_k^+)^2} - \frac{1}{(\xi_k^-)^2} \right] \right\} \\ &\simeq \frac{\Delta_0^2}{\pi^2} \frac{4}{3} \int_0^\infty k^2 dk \frac{8\lambda^4 k^2}{(k^2 + E_B)[(k^2 + E_B)^2 - 4\lambda^2 k^2]^2}. \end{aligned} \quad (80)$$

Comparing the above results with the equation for the molecule effective mass, we find that $n_\lambda/n \simeq 1 - 2m/m_B$. Therefore, at large SOC, the superfluid density is suppressed by a factor $2m/m_B$, i.e.,

$$n_s \simeq \frac{2m}{m_B} n. \quad (81)$$

For $\lambda \rightarrow \infty$, using the result for $2m/m_B$ at $\kappa = 0$, we find that the ratio n_s/n approaches a universal value,

$$\frac{n_s}{n}(\lambda/k_F \rightarrow \infty) \rightarrow \frac{14}{3(4 + \sqrt{2})} = 0.862. \quad (82)$$

To further understand this result, we consider the effective action for the phase field $\theta(x)$. To this end, we write the order parameter as $\Phi(x) = \Delta(x)e^{i\theta(x)}$. In the static limit, we can obtain the effective Hamiltonian for the phase field, $H_{\text{eff}} = (J_s/2) \int d^3\mathbf{r} [\nabla\theta(\mathbf{r})]^2$, where the superfluid phase stiffness J_s is related to the superfluid density n_s by $J_s = n_s/(4m)$. Therefore, at large SOC, we have

$$J_s \simeq \frac{2m}{m_B} \frac{n}{4m} = \frac{n_B}{m_B}, \quad (83)$$

where $n_B = n/2$ is the density of bosons (rashbons). This means that, at large SOC, the superfluid phase stiffness self-consistently recovers that for a rashbon gas with a non-trivial effective mass m_B . We emphasize that this interesting result was first observed by us in 2D Fermi gases with Rashba spin-orbit coupling [25].

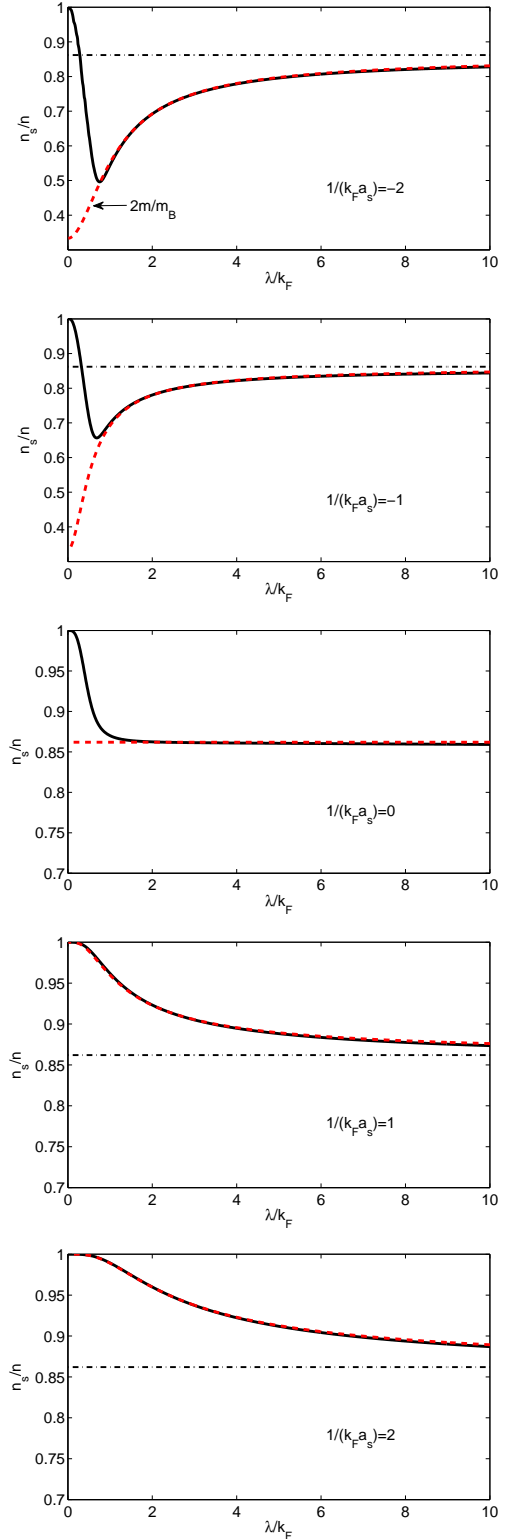


FIG. 7: The superfluid density n_s (divided by n) as a function of λ/k_F for various values of $1/(k_F a_s)$. The red dashed line shows the analytical result $2m/m_B$. The dot-dashed line corresponds to the value 0.862 for $\lambda a_s \rightarrow \infty$.

This result also indicates that the Galilean invariance, which is explicitly broken in the original fermion Hamiltonian, can be effectively restored at the boson (rashbon) level at large SOC. This is due to the fact that at large SOC the system becomes a weakly interacting Bose-Einstein condensate of non-relativistic rashbons which have a non-trivial effective mass m_B . We will show this conclusion explicitly in the next section by deriving the Gross-Pitaevskii free energy for the dilute rashbon condensate at large SOC.

(C) Numerical Results. The superfluid density at zero temperature can be expressed in terms of the dimensionless parameters as

$$\frac{n_s}{n} = 1 - \frac{g_1}{2} \sum_{\alpha=\pm} \alpha \int_0^\infty z dz \frac{z^2 + 2\alpha g_1 z - x_1 + \frac{x_2^2}{z^2 - x_1}}{\sqrt{(z^2 + 2\alpha g_1 z - x_1)^2 + x_2^2}}. \quad (84)$$

It can be numerically obtained using the solutions of x_1 and x_2 from the gap and number equations.

The numerical results for n_s/n as a function of λ/k_F for different values of $1/(k_F a_s)$ are shown in Fig. 7. For negative values or small positive values of $1/(k_F a_s)$, the numerical result becomes in good agreement with the analytical result $n_s/n \simeq 2m/m_B$ when $\lambda/k_F > 1$, which is consistent with the observation that the system enters the rashbon BEC regime at $\lambda/k_F \sim 1$. For large positive values of $1/(k_F a_s)$ (in fact even for $1/(k_F a_s) = 1$), the numerical results are always in good agreement with the analytical result for all values of λ/k_F .

For both negative and positive values of $1/(k_F a_s)$, we find that n/n_s approaches a universal value 0.862 when $\lambda/k_F \rightarrow \infty$, as indicated from the analytical observation.

V. BOSE-EINSTEIN CONDENSATION OF WEAKLY INTERACTING RASHBONS

As we have shown in the last section, the superfluid state in the large SOC limit is a Bose-Einstein condensation of rashbons. We are interested in the interactions among the rashbons. In this section, we will derive the Gross-Pitaevskii free energy for a dilute rashbon condensate, which allow us to extract the rashbon-rashbon scattering length. Another goal of this section is to show that the Galilean invariance, which is explicitly broken in the original fermion Hamiltonian, can be effectively recovered at the boson (rashbon) level at large SOC.

To this end, we consider the mean field theory where the auxiliary boson field $\Phi(x)$ is replaced by its expectation value $\langle \Phi(x) \rangle = \Delta(x)$. In the large SOC limit $\lambda \rightarrow \infty$, the fermion chemical potential μ approaches $-E_B/2$. Since the pairing gap $|\Delta| \ll |\mu|$, we can expand the effective action in powers of $|\Delta|$ (as well as in powers of its space-time derivatives), which results in a Ginzburg-Landau free energy functional

$$V_{GL}[\Delta(x)] = \int dx \left[\Delta^\dagger(x) \left(a \frac{\partial}{\partial \tau} - b \nabla^2 \right) \Delta(x) + c |\Delta(x)|^2 + \frac{1}{2} d |\Delta(x)|^4 \right]. \quad (85)$$

A. Calculation of the Ginzburg-Landau Coefficients

The coefficients c and d of the potential terms can be obtained from the mean field thermodynamic potential $\Omega_0 = (T/V) \mathcal{S}_{\text{eff}}[\Delta^\dagger, \Delta]$ which can be evaluated as

$$\Omega = -\frac{|\Delta|^2}{4\pi a_s} - \sum_{\mathbf{k}} \left(\frac{E_{\mathbf{k}}^+ + E_{\mathbf{k}}^-}{2} - \xi_{\mathbf{k}} - \frac{|\Delta|^2}{2\epsilon_{\mathbf{k}}} \right). \quad (86)$$

We have

$$c = \frac{\partial \Omega}{\partial |\Delta|^2} \Big|_{\Delta=0}, \quad d = \frac{\partial^2 \Omega}{\partial (|\Delta|^2)^2} \Big|_{\Delta=0}. \quad (87)$$

After a simple algebra, the coefficients α and β can be evaluated as

$$c = \frac{1}{4\pi} \left(\frac{-2\mu - 2\lambda^2}{\sqrt{-2\mu - \lambda^2}} - \frac{1}{a_s} \right), \quad d = \frac{1}{16\pi} \frac{-2\mu + 2\lambda^2}{(-2\mu - \lambda^2)^{5/2}}. \quad (88)$$

From the expression of c , we find that a quantum phase transition from vacuum to Bose condensation takes place at $\mu = -E_B/2$. Thus near the phase transition, c can be simplified as

$$c \simeq -\frac{1}{8\pi} \frac{E_B}{(E_B - \lambda^2)^{3/2}} \mu_B, \quad (89)$$

where $\mu_B = 2\mu + E_B \ll E_B$ is the boson chemical potential. Further, setting $\mu = -E_B/2$, d can be reduced to

$$d \simeq \frac{1}{16\pi} \frac{E_B + 2\lambda^2}{(E_B - \lambda^2)^{5/2}}. \quad (90)$$

The coefficients a and b of the kinetic terms can be obtained from the inverse boson propagator $\mathcal{D}^{-1}(Q)$ with $\Delta = 0$. It can be evaluated as

$$\mathcal{D}^{-1}(Q) = \frac{1}{U} - \frac{1}{4} \sum_{\alpha, \gamma=\pm} \sum_{\mathbf{k}} \frac{1 + \alpha \gamma \mathcal{T}_{\mathbf{k}\mathbf{q}}}{\xi_{\mathbf{k}+\mathbf{q}/2}^\alpha + \xi_{\mathbf{k}-\mathbf{q}/2}^\gamma - i\nu_n} \quad (91)$$

In the large SOC limit, the coefficients a and b can be obtained by the small momentum expansion for $\mathcal{D}^{-1}(Q)$. We have

$$\mathcal{D}^{-1}(Q) \simeq -a \left(i\nu_n + \mu_B - \frac{\mathbf{q}^2}{2m_B} \right), \quad (92)$$

where m_B is the rashbon effective mass determined by (28), and a is given by

$$a = \frac{1}{8\pi} \frac{E_B}{(E_B - \lambda^2)^{3/2}}. \quad (93)$$

We observe the relation $c = \mathcal{D}^{-1}(0) = -a\mu_B$.

B. Gross-Pitaevskii Free Energy

According to the above results for the Ginzburg-Landau coefficients, if we define the new condensate wave function $\psi(x)$ by

$$\psi(x) = \sqrt{a}\Delta(x), \quad (94)$$

the Ginzburg-Landau free energy can be reduced to the Gross-Pitaevskii free energy of a dilute Bose gas,

$$V_{\text{GP}}[\psi(x)] = \int dx \left[\psi^\dagger(x) \left(\frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m_B} \right) \psi(x) - \mu_B |\psi(x)|^2 + \frac{1}{2} \frac{4\pi a_{\text{BB}}}{m_B} |\psi(x)|^4 \right], \quad (95)$$

where a_{BB} is the boson-boson scattering length. Its explicit expression is

$$a_{\text{BB}} = m_B \frac{E_B + 2\lambda^2}{E_B^2} \sqrt{E_B - \lambda^2}. \quad (96)$$

Note that $m = 1$ in our units. For $\lambda = 0$ and $a_s > 0$, using the result $m_B = 2$ and $E_B = 1/a_s^2$, we recover the well-known result $a_{\text{BB}} = 2a_s$ [5]. One remark here is that this result is the mean field result which is not exact. In the absence of SOC, exact four-body calculation shows that $a_{\text{BB}} \simeq 0.6a_s$ [35]. Therefore, it is interesting to explore the exact rashbon-rashbon scattering length in the future studies. Another theoretical framework to obtain more exact a_{BB} is to include the Gaussian fluctuations [36].

The Gross-Pitaevskii free energy explicitly shows that the Galilean invariance, which is explicitly broken in the original fermion Hamiltonian, can be effectively recovered at the boson (rashbon) level at large SOC.

C. Rashbon-Rashbon Scattering Length

Using the expressions for the binding energy E_B and the effective mass m_B , we obtain

$$a_{\text{BB}} = \frac{1}{\lambda} \frac{2(\mathcal{J} + 2)\sqrt{\mathcal{J} - 1}}{\mathcal{J}^2 \left[\frac{7}{3} - \frac{4}{3} \left(\frac{\mathcal{J} - 1}{\mathcal{J}} \right)^{3/2} - \frac{2}{\mathcal{J}} \right]}. \quad (97)$$

We find that the quantity λa_{BB} depends only on the dimensionless parameter $\kappa = 1/(\lambda a_s)$. For the case $\lambda a_s \rightarrow \infty$ or $\kappa = 0$, we have $\mathcal{J} = 2$ and therefore

$$a_{\text{BB}}(\lambda a_s \rightarrow \infty) = \frac{1}{\lambda} \frac{3(4 + \sqrt{2})}{7} = \frac{2.32}{\lambda}. \quad (98)$$

The numerical result for the scattering length a_{BB} is shown in Fig. 8. We find that the quantity λa_{BB} has a maximum near the point $\kappa = 0$, at $\kappa = -2.11$.

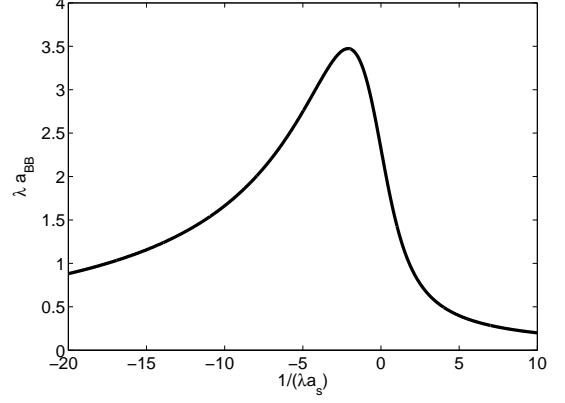


FIG. 8: The molecule scattering length a_{BB} (divided by $1/\lambda$) as a function of the dimensionless parameter $\kappa = 1/(\lambda a_s)$.

D. Rashbon Chemical Potential

For a uniform system, the expectation value of the condensate $\psi(x)$ should be determined by minimizing the Gross-Pitaevskii free energy. We find that the minimum is given by

$$|\psi_0|^2 = \frac{\mu_B}{g_0}, \quad (99)$$

where $g_0 = 4\pi a_{\text{BB}}/m_B$. The total density of the bosons is $n_B = n/2 = |\psi_0|^2 = a\Delta_0^2$. Therefore, the boson chemical potential can be given by

$$\mu_B = \frac{2\pi a_{\text{BB}}}{m_B} n. \quad (100)$$

For the case $\lambda a_s \rightarrow \infty$, using the result for m_B and a_{BB} , we obtain

$$\mu_B(\lambda a_s \rightarrow \infty) = \frac{2\pi n}{\lambda}. \quad (101)$$

VI. GAUSSIAN FLUCTUATION AND COLLECTIVE EXCITATIONS

To study the collective excitations, we consider the fluctuations around the mean field. Making the field shift $\Phi(x) \rightarrow \Delta_0 + \phi(x)$, we can expand the effective action \mathcal{S}_{eff} in powers of the fluctuations. The zeroth order term $\mathcal{S}_{\text{eff}}^{(0)}$ is just the mean field result, and the linear terms vanish automatically guaranteed by the saddle point condition for Δ_0 . The quadratic terms, corresponding to Gaussian fluctuations, can be evaluated as

$$\mathcal{S}_{\text{eff}}^{(2)}[\phi, \phi^\dagger] = \frac{1}{2} \sum_Q \begin{pmatrix} \phi^\dagger(Q) & \phi(-Q) \end{pmatrix} \mathbf{M}(Q) \begin{pmatrix} \phi(Q) \\ \phi^\dagger(-Q) \end{pmatrix} \quad (102)$$

where the inverse boson propagator \mathbf{M} takes the form

$$\mathbf{M}(Q) = \begin{pmatrix} \mathbf{M}_{11}(Q) & \mathbf{M}_{12}(Q) \\ \mathbf{M}_{21}(Q) & \mathbf{M}_{22}(Q) \end{pmatrix} \quad (103)$$

with the relations $\mathbf{M}_{11}(Q) = \mathbf{M}_{22}(-Q)$ and $\mathbf{M}_{12}(Q) = \mathbf{M}_{21}(Q)$. The matrix elements of $\mathbf{M}(Q)$ can be expressed in terms of the fermion propagator $\mathcal{G}(K)$. We have

$$\begin{aligned}\mathbf{M}_{11}(Q) &= \frac{1}{U} - \frac{1}{2} \sum_K \text{Tr} [\mathcal{G}_{11}(K+Q) \tau_y \mathcal{G}_{22}(K) \tau_y], \\ \mathbf{M}_{12}(Q) &= \frac{1}{2} \sum_K \text{Tr} [\mathcal{G}_{12}(K+Q) \tau_y \mathcal{G}_{12}(K) \tau_y].\end{aligned}\quad (104)$$

At zero temperature, the explicit form of $\mathbf{M}(Q)$ can be evaluated as

$$\mathbf{M}_{11}(Q) = \frac{1}{U} + \frac{1}{4} \sum_{\alpha, \gamma = \pm} \sum_{\mathbf{k}} \left[\frac{(u_{\mathbf{k}+\mathbf{q}/2}^\alpha)^2 (u_{\mathbf{k}-\mathbf{q}/2}^\gamma)^2}{i\nu_n - E_{\mathbf{k}+\mathbf{q}/2}^\alpha - E_{\mathbf{k}-\mathbf{q}/2}^\gamma} - \frac{(v_{\mathbf{k}+\mathbf{q}/2}^\alpha)^2 (v_{\mathbf{k}-\mathbf{q}/2}^\gamma)^2}{i\nu_n + E_{\mathbf{k}+\mathbf{q}/2}^\alpha + E_{\mathbf{k}-\mathbf{q}/2}^\gamma} \right] (1 + \alpha\gamma \mathcal{T}_{\mathbf{k}\mathbf{q}}) \quad (105)$$

and

$$\mathbf{M}_{12}(Q) = \frac{1}{4} \sum_{\alpha, \gamma = \pm} \sum_{\mathbf{k}} \left[\frac{u_{\mathbf{k}+\mathbf{q}/2}^\alpha v_{\mathbf{k}+\mathbf{q}/2}^\alpha u_{\mathbf{k}-\mathbf{q}/2}^\gamma v_{\mathbf{k}-\mathbf{q}/2}^\gamma}{i\nu_n + E_{\mathbf{k}+\mathbf{q}/2}^\alpha + E_{\mathbf{k}-\mathbf{q}/2}^\gamma} - \frac{u_{\mathbf{k}+\mathbf{q}/2}^\alpha v_{\mathbf{k}+\mathbf{q}/2}^\alpha u_{\mathbf{k}-\mathbf{q}/2}^\gamma v_{\mathbf{k}-\mathbf{q}/2}^\gamma}{i\nu_n - E_{\mathbf{k}+\mathbf{q}/2}^\alpha - E_{\mathbf{k}-\mathbf{q}/2}^\gamma} \right] (1 + \alpha\gamma \mathcal{T}_{\mathbf{k}\mathbf{q}}). \quad (106)$$

Here the BCS distribution functions are defined as $(v_{\mathbf{k}}^\alpha)^2 = (1 - \xi_{\mathbf{k}}^\alpha/E_{\mathbf{k}}^\alpha)/2$ and $(u_{\mathbf{k}}^\alpha)^2 = (1 + \xi_{\mathbf{k}}^\alpha/E_{\mathbf{k}}^\alpha)/2$. In the absence of SOC, $\lambda = 0$, the expressions for $\mathbf{M}_{11}(Q)$ and $\mathbf{M}_{12}(Q)$ recover the results obtained in [5].

A. Bogoliubov Excitation in the Rashbon Condensate

At large SOC and/or attraction, the superfluid state is a Bose-Einstein condensation of weakly interacting Bose gas. Thus we expect that the low energy collective excitation in this case recover the well known Bogoliubov excitation spectrum in a weakly interacting Bose condensate [37]. In this part, we will give an explicit proof for this.

In the large SOC and/or strong coupling limit, the chemical potential reads $\mu \simeq -E_B/2$ and we have $\Delta_0 \ll |\mu|$. In this case, we can expand the matrix elements of \mathbf{M} in powers of $\Delta_0/|\mu|$ and keep only the leading-order terms. Following this spirit, we obtain

$$\begin{aligned}\mathbf{M}_{11}(Q) &\simeq \mathcal{D}^{-1}(Q) + X\Delta_0^2, \\ \mathbf{M}_{12}(Q) &\simeq Y\Delta_0^2,\end{aligned}\quad (107)$$

where the coefficients X and Y are given by

$$X = 2Y = \frac{1}{4} \sum_{\mathbf{k}} \left[\frac{1}{(\xi_{\mathbf{k}}^+)^3} + \frac{1}{(\xi_{\mathbf{k}}^-)^3} \right] = 2d. \quad (108)$$

Further, taking the small momentum expansion for $\mathcal{D}^{-1}(Q)$, we obtain

$$\mathbf{M}_{11}(Q) \simeq -a \left(i\nu_n + \mu_B - \frac{\mathbf{q}^2}{2m_B} \right) + 2d\Delta_0^2. \quad (109)$$

Therefore, in the large SOC and/or strong coupling limit, the

boson propagator $\mathbf{M}(Q)$ can be well approximated by

$$\begin{aligned}\mathbf{M}_{11}(Q) &\simeq a \left(-i\nu_n + \frac{\mathbf{q}^2}{2m_B} - \mu_B + 2g_0|\psi_0|^2 \right) \\ \mathbf{M}_{22}(Q) &\simeq a \left(i\nu_n + \frac{\mathbf{q}^2}{2m_B} - \mu_B + 2g_0|\psi_0|^2 \right) \\ \mathbf{M}_{12}(Q) &= \mathbf{M}_{21}(Q) \simeq ag_0|\psi_0|^2,\end{aligned}\quad (110)$$

where $g_0 = 4\pi a_{BB}/m_B$ and $|\psi_0|^2 = \mu_B/g_0$ is the minimum of the Gross-Pitaevskii free energy (corresponding to the saddle point Δ_0 of the effective potential). From the Gross-Pitaevskii free energy, the boson density reads $n_B = n/2 = |\psi_0|^2$. Utilizing these results, we obtain

$$\mathbf{M}(Q) \simeq a \begin{pmatrix} -i\nu_n + \frac{\mathbf{q}^2}{2m_B} + g_0 n_B & g_0 n_B \\ g_0 n_B & i\nu_n + \frac{\mathbf{q}^2}{2m_B} + g_0 n_B \end{pmatrix}, \quad (111)$$

Taking the analytical continuation $i\nu_n \rightarrow \omega + i0^+$, the dispersion $\omega = \omega(\mathbf{q})$ of the collective mode is obtained by solving the equation

$$\det \mathbf{M}[\mathbf{q}, \omega(\mathbf{q})] = 0. \quad (112)$$

Therefore, the Goldstone mode takes a dispersion relation given by

$$\omega(\mathbf{q}) = \sqrt{\frac{\mathbf{q}^2}{2m_B} \left(\frac{\mathbf{q}^2}{2m_B} + \frac{8\pi a_{BB} n_B}{m_B} \right)}. \quad (113)$$

This is just the Bogoliubov excitation spectrum in a dilute Bose condensate where the bosons possess a mass m_B and a two-body scattering length a_{BB} .

B. Collective Modes in the BCS-BEC Crossover

The dispersions of the collective modes are in principle determined by the equation $\det \mathbf{M}[\mathbf{q}, \omega(\mathbf{q})] = 0$. To make the result more physical, we decompose the complex fluctuation field $\phi(x)$ into its amplitude mode $\lambda(x)$ and phase mode $\theta(x)$, $\phi(x) = \lambda(x) + i\theta(x)$. Then the fluctuation part of the effective action takes the form

$$\mathcal{S}_{\text{eff}}^{(2)} = \frac{1}{2} \sum_{\mathbf{Q}} \begin{pmatrix} \lambda^*(\mathbf{Q}) & \theta^*(\mathbf{Q}) \end{pmatrix} \mathbf{N}(\mathbf{Q}) \begin{pmatrix} \lambda(\mathbf{Q}) \\ \theta(\mathbf{Q}) \end{pmatrix}, \quad (114)$$

where the matrix $\mathbf{N}(\mathbf{Q})$ is defined as

$$\mathbf{N}(\mathbf{Q}) = 2 \begin{pmatrix} \mathbf{M}_{11}^+ + \mathbf{M}_{12} & i\mathbf{M}_{11}^- \\ -i\mathbf{M}_{11}^- & \mathbf{M}_{11}^+ - \mathbf{M}_{12} \end{pmatrix}. \quad (115)$$

Here the quantities \mathbf{M}_{11}^\pm are defined as

$$\mathbf{M}_{11}^\pm(\mathbf{q}, \omega) = \frac{1}{2} [\mathbf{M}_{11}(\mathbf{q}, \omega) \pm \mathbf{M}_{11}(\mathbf{q}, -\omega)]. \quad (116)$$

We notice that \mathbf{M}_{11}^+ and \mathbf{M}_{11}^- are even and odd functions of ω , respectively.

From the explicit form of $\mathbf{M}_{11}(\mathbf{Q})$, we have $\mathbf{M}_{11}^-(\mathbf{q}, 0) = 0$. Therefore the amplitude and phase modes decouple completely at $\omega = 0$. Furthermore, using the saddle point condition for the order parameter Δ_0 , we find $\mathbf{M}_{11}^+(\mathbf{0}, 0) = \mathbf{M}_{12}(\mathbf{0}, 0)$, which ensures that the phase mode at $\mathbf{q} = \mathbf{0}$ is gapless, i.e., the Goldstone mode.

We now determine the velocity c_s of the Goldstone mode, $\omega(\mathbf{q}) = c_s |\mathbf{q}|$ for $\omega, |\mathbf{q}| \ll \min_{\mathbf{k}} \{E_{\mathbf{k}}^\pm\}$. For this purpose, we make a small \mathbf{q} and ω expansion of $\mathbf{N}(\mathbf{Q})$,

$$\begin{aligned} \mathbf{M}_{11}^+ + \mathbf{M}_{12} &= A + C|\mathbf{q}|^2 - D\omega^2 + \dots, \\ \mathbf{M}_{11}^+ - \mathbf{M}_{12} &= Q|\mathbf{q}|^2 - R\omega^2 + \dots, \\ \mathbf{M}_{11}^- &= -B\omega + \dots. \end{aligned} \quad (117)$$

Here we note that the coefficient Q is proportional to the superfluid density n_s and the superfluid phase stiffness J_s . The explicit form of A , B , D , R and Q can be calculated as

$$\begin{aligned} A &= \frac{1}{4} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{\Delta_0^2}{(E_{\mathbf{k}}^\alpha)^3}, \\ B &= \frac{1}{8} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{\xi_{\mathbf{k}}^\alpha}{(E_{\mathbf{k}}^\alpha)^3}, \\ D &= \frac{1}{16} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \left[\frac{1}{(E_{\mathbf{k}}^\alpha)^3} - \frac{\Delta_0^2}{(E_{\mathbf{k}}^\alpha)^5} \right], \\ R &= \frac{1}{16} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{1}{(E_{\mathbf{k}}^\alpha)^3}, \\ Q &= \frac{J_s}{2\Delta_0^2} = \frac{n_s}{8m\Delta_0^2}. \end{aligned} \quad (118)$$

The Goldstone mode velocity or the so-called sound velocity in the superfluid state is given by

$$c_s = \sqrt{\frac{Q}{B^2/A + R}}. \quad (119)$$

The corresponding eigenvector of \mathbf{N} is $(\lambda, \theta) = (-ic|\mathbf{q}|B/A, 1)$, which is a pure phase mode at $\mathbf{q} = \mathbf{0}$ but has an admixture of the amplitude mode controlled by B at finite \mathbf{q} . Another massive mode, or the so-called Anderson-Higgs mode, has a mass gap

$$M_g = \sqrt{\frac{B^2 + AR}{DR}}. \quad (120)$$

(A) Analytical Results for Large SOC. In the rashbon BEC limit $\lambda/k_F \gg 1$, we have $\mu \simeq E_B/2$ and $\Delta_0 \ll |\mu|$. Therefore, the coefficients A , B , D , R and Q can be well approximated as

$$\begin{aligned} A &\simeq \frac{\Delta_0^2}{4} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{1}{(\xi_{\mathbf{k}}^\alpha)^3} = 2\Delta_0^2 d, \\ B &\simeq \frac{1}{8} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{1}{(\xi_{\mathbf{k}}^\alpha)^2} = a, \\ D &\simeq \frac{1}{16} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{1}{(\xi_{\mathbf{k}}^\alpha)^3} = \frac{d}{2}, \\ R &\simeq \frac{1}{16} \sum_{\alpha=\pm} \sum_{\mathbf{k}} \frac{1}{(\xi_{\mathbf{k}}^\alpha)^3} = \frac{d}{2}, \\ Q &\simeq \frac{1}{2\Delta_0^2} \frac{n_B}{m_B}. \end{aligned} \quad (121)$$

In this case, we find that $B^2/A \gg R$ and therefore the amplitude and phase modes are strongly coupled.

The sound velocity c_s and the mass gap M_g read

$$\begin{aligned} c_s &= \sqrt{\frac{AQ}{B^2}} \simeq \sqrt{\frac{d}{a^2} \frac{n_B}{m_B}}, \\ M_g &= \sqrt{\frac{B^2}{DR}} \simeq \frac{2a}{d}. \end{aligned} \quad (122)$$

Using the relation $d/a^2 = 4\pi a_{BB} n_B / m_B$, the sound velocity recovers the result for a weakly interacting rashbon gas,

$$c_s = \sqrt{\frac{4\pi a_{BB} n_B}{m_B^2}} = \sqrt{\frac{\mu_B}{m_B}}. \quad (123)$$

Therefore, in the BEC limit, the quantity c_s/c_0 depends only on the dimensionless parameter $\kappa = 1/(\lambda a_s)$, where $c_0 = \sqrt{2\pi n/\lambda}$. Using the results for m_B and a_{BB} , we obtain

$$c_s = c_0 \sqrt{\frac{(\mathcal{J}+2)\sqrt{\mathcal{J}-1}}{2\mathcal{J}^2} \left[\frac{7}{3} - \frac{4}{3} \left(\frac{\mathcal{J}-1}{\mathcal{J}} \right)^{3/2} - \frac{2}{\mathcal{J}} \right]}. \quad (124)$$

The numerical result for the quantity c/c_0 is shown in Fig. 9. We find that it has a maximum near the point $\kappa = 0$, at $\kappa = -0.18$. For the case $\lambda a_s \rightarrow \infty$, we have

$$c_s(\lambda a_s \rightarrow \infty) = c_0 \sqrt{\frac{7}{3(4 + \sqrt{2})}} = 0.66c_0. \quad (125)$$

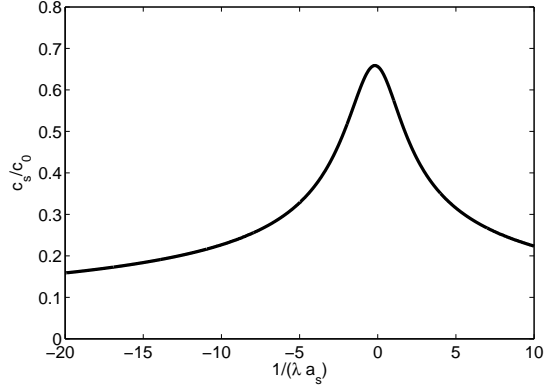


FIG. 9: The sound velocity c_s of the Goldstone mode (divided by c_0) in the RBEC regime as a function of the dimensionless parameter $\kappa = 1/(\lambda a_s)$.

Using the expressions for a and d , we obtain the explicit form of the mass gap M_g ,

$$M_g = \frac{4E_B(E_B - \lambda^2)}{E_B + 2\lambda^2} = \lambda^2 \frac{4\mathcal{J}(\mathcal{J} - 1)}{\mathcal{J} + 2}. \quad (126)$$

Therefore, in the BEC limit, the quantity M_g/λ^2 depends only on the dimensionless parameter $\kappa = 1/(\lambda a_s)$. The numerical result is shown in Fig. 10. We find that it is very small in the limit $\kappa \rightarrow -\infty$, and increases rapidly in the regime $\kappa > 0$. For the case $\lambda a_s \rightarrow \infty$, we have $E_B = 2\lambda^2$ and therefore

$$M_g(\lambda a_s \rightarrow \infty) = 2\lambda^2. \quad (127)$$

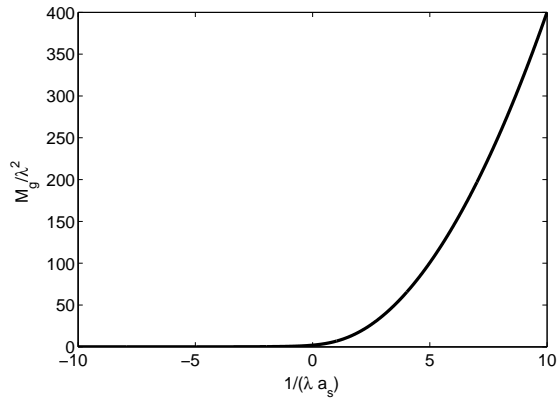


FIG. 10: The mass gap M_g of the Anderson-Higgs mode (divided by λ^2) in the RBEC regime as a function of the dimensionless parameter $\kappa = 1/(\lambda a_s)$.

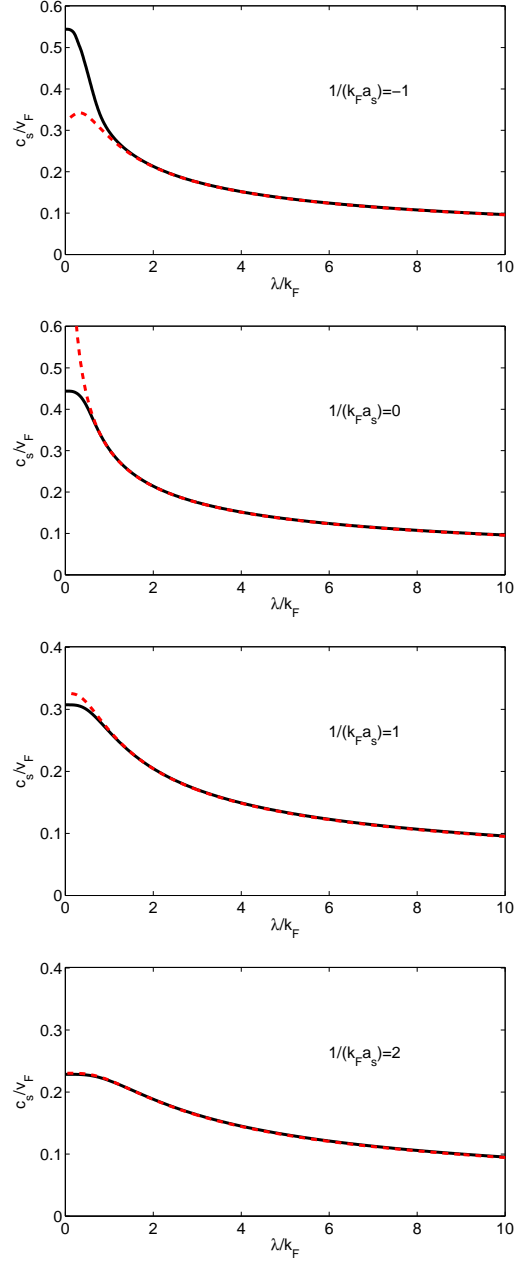


FIG. 11: The velocity of the Goldstone mode (sound velocity) c_s (divided by v_F) as a function of λ/k_F for various values of $1/k_F a_s$. The red dashed lines corresponds to the analytical result (124).

(B) Numerical Results. Using the same trick in Section IV,

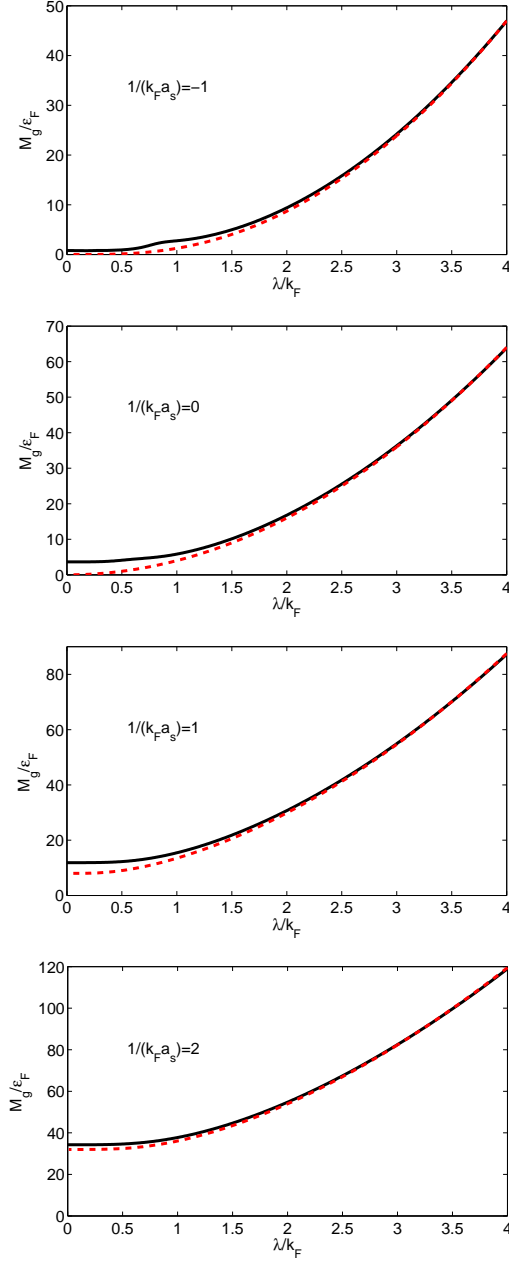


FIG. 12: The mass gap M_g of the Anderson-Higgs mode (divided by ϵ_F) as a function of λ/k_F for various values of $1/(k_F a_s)$. The red dashed lines corresponds to the analytical result (126).

we obtain

$$\begin{aligned}
 A &= \frac{\Delta_0^2}{4\pi^2} \int_0^\infty dk \frac{k^2 + \lambda^2}{[(\epsilon_k - \tilde{\mu})^2 + \Delta_0^2]^{3/2}} \equiv \frac{k_F}{2\pi^2} \tilde{A}, \\
 B &= \frac{1}{8\pi^2} \int_0^\infty dk \frac{(k^2 + \lambda^2)(\epsilon_k - \tilde{\mu})}{[(\epsilon_k - \tilde{\mu})^2 + \Delta_0^2]^{3/2}} \equiv \frac{1}{2\pi^2 k_F} \tilde{B}, \\
 D &= \frac{1}{16\pi^2} \int_0^\infty dk \frac{(k^2 + \lambda^2)(\epsilon_k - \tilde{\mu})^2}{[(\epsilon_k - \tilde{\mu})^2 + \Delta_0^2]^{5/2}} \equiv \frac{1}{2\pi^2 k_F^3} \tilde{D}, \\
 R &= \frac{1}{16\pi^2} \int_0^\infty dk \frac{k^2 + \lambda^2}{[(\epsilon_k - \tilde{\mu})^2 + \Delta_0^2]^{3/2}} \equiv \frac{1}{2\pi^2 k_F^3} \tilde{R}, \\
 Q &\equiv \frac{1}{2\pi^2 k_F} \tilde{Q},
 \end{aligned} \tag{128}$$

where the dimensionless quantities \tilde{A} , \tilde{B} , \tilde{D} , \tilde{R} and \tilde{Q} are defined as

$$\begin{aligned}
 \tilde{A} &= x_2^2 \int_0^\infty dz \frac{z^2 + g_1^2}{[(z^2 - g_1^2 - x_1)^2 + x_2^2]^{3/2}}, \\
 \tilde{B} &= \int_0^\infty dz (z^2 + g_1^2) \frac{z^2 - g_1^2 - x_1}{[(z^2 - g_1^2 - x_1)^2 + x_2^2]^{3/2}}, \\
 \tilde{D} &= \int_0^\infty dz (z^2 + g_1^2) \frac{(z^2 - g_1^2 - x_1)^2}{[(z^2 - g_1^2 - x_1)^2 + x_2^2]^{5/2}}, \\
 \tilde{R} &= \int_0^\infty dz \frac{z^2 + g_1^2}{[(z^2 - g_1^2 - x_1)^2 + x_2^2]^{3/2}}, \\
 \tilde{Q} &= \frac{1}{3x_2^2} - \frac{g_1}{6x_2^2} \sum_{\alpha=\pm} \alpha \int_0^\infty z dz \frac{z^2 + 2\alpha g_1 z - x_1 + \frac{x_2^2}{z^2 - x_1}}{\sqrt{(z^2 + 2\alpha g_1 z - x_1)^2 + x_2^2}}.
 \end{aligned} \tag{129}$$

Therefore, we have

$$\frac{c_s}{v_F} = \sqrt{\frac{\tilde{Q}}{\tilde{B}^2/\tilde{A} + \tilde{R}}} \tag{130}$$

and

$$\frac{M_g}{\epsilon_F} = 2 \sqrt{\frac{\tilde{B}^2 + \tilde{A}\tilde{R}}{\tilde{D}\tilde{R}}}, \tag{131}$$

where $v_F = k_F/m$ ($m = 1$) is the Fermi velocity for the non-interacting Fermi gas in the absence of SOC.

Using the solutions of x_1 and x_2 from the gap and number equations, we can calculate the quantity c_s/v_F and M_g/ϵ_F for given values of $1/(k_F a_s)$ and λ/k_F . The numerical results are shown in Fig. 11 and Fig. 12. For large negative values of $1/(k_F a_s)$ and $\lambda/k_F \rightarrow 0$, we recover the well-known result $c_s = v_F/\sqrt{3}$ for weak coupling fermionic superfluids [5]. For negative values or small positive values of $1/(k_F a_s)$, the numerical result becomes already in good agreement with the analytical results (124) and (126) at $\lambda/k_F \sim 1$, which is consistent with the observation that the system enters the rashbon BEC regime at $\lambda/k_F \sim 1$. For large positive values of $1/(k_F a_s)$, the numerical results are in good agreement with the analytical results for all values of λ/k_F .

For very large λ/k_F , we find that the numerical results fit very well with the following scaling behavior

$$\frac{c_s}{v_F} = 0.66 \sqrt{\frac{2\pi}{3}} \left(\frac{\lambda}{k_F} \right)^{-1/2}, \quad \frac{M_g}{\epsilon_F} = 4 \left(\frac{\lambda}{k_F} \right)^2, \quad (132)$$

for both negative and positive values of $1/(k_F a_s)$, as indicated from the analytical observations.

VII. SUMMARY

In summary, we have presented a comprehensive study of the BCS-BEC crossover problem in 3D Fermi gases with a spherical spin-orbit coupling which can be realized by a 3D symmetrical configuration of the synthetic SU(2) gauge field. The two-body problem, the superfluid ground-state properties, and the behaviors of collective excitations are studied. Ana-

lytical results and interesting universal behaviors for various physical quantities at large SOC are obtained. We notice that there has been experimental proposal for the realization of 3D spherical spin-orbit coupling for cold fermionic atoms [38]. Therefore, it is interesting to test our theoretical predictions in future experiments of cold Fermi gases with 3D spherical spin-orbit coupling.

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Note Added — During the preparation of this manuscript, we became aware of the recent paper by Vyasanakere and Shenoy [39], where similar results were reported.

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